HIGHER STRUCTURES IN HOMOTOPY THEORY 2018

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1. July 2: Emily Riehl: The model-independent theory of $(\infty, 1)$-categories I

I was a student at Cambridge 10 years ago and here is where I met my collaborator on this work. Everything I'm going to say is joint with Dominic Verity, a Cambridge alum now at Macquarie. There are references listed for this, a Cliff's Notes version and a longer version, so today is chapter 2 of the book draft we're writing, "Elements of ∞-category theory"

This program is about higher structures in homotopy theory. What the higher structures in homotopy theory usually assemble into is an $(\infty, 1)$-category. You have objects and morphisms and then there are homotopies which are data, between morphisms, and then between the higher morphisms and so on. So morphisms are paths in some space and these have a kind of composition which is not strictly defined, not strictly associative, not strictly unital. There's some invertibility and when you forget the higher stuff you drop to regular 1-category theory.

If you're interested in working here then you want to have the familiar theorems of 1-category theory, and the aim is to develop the theory of ∞-categories, extending 1-categories. I've made a schematic definition, not a precise one. One uses a variety of models to make this precise, developed over the past couple of decades by several mathematicians, including ones in this room. The idea going back at least to Toën but probably much further, is that the model shouldn't matter, so things are *model-independent*.

So one other thing I wanted to say is equally people who have not yet made the acquaintance of ∞-categories and people with a background as well, because there is a difference in attitude.

The difference, the main difference between our approach as opposed to previous work (to which we are indebted) is that it is "synthetic" rather than "analytic." Joyal and Lurie are often made in quasi-categories, relative to the combinatorics of simplicial sets. For example, you do induction on simplices. You don't state things in reference to a particular model in a synthetic approach, it's axiomatic, and then they will be true in any model.

Necessarily if we're developing a synthetic approach it's important that this work specializes to the Joyal–Lurie theory of quasi-categories. Everybody agrees that the definition Joyal gave for limit, that Lurie gave for adjunction, are correct, and rest assured that if we specialize to quasi-categories, we recover those.

I'm not saying that the synthetic is better than analytic—there are more things that you can prove in coordinates than without them, but they are complementary.

The third quality is that these are invariant under change of model. The hope is that theorems proven for quasi-categories apply equally well under change of context. Toën has a result relating the homotopy theory of $(\infty,1)$-categories, so maybe we think that some of this is evident. But some people are redoing categorical things for complete Segal spaces, and so we assume it's not obvious that these things work.

So if you have a diagram in one model and move it to another model and find a limit, was there a limit in the original model?

Another point is that this is as simple as possible. We want, one day, to allow undergraduates to learn this without twisting themselves into a knot. The axioms will describe the $(\infty,2)$-category where our models work, and this is a very strict category. None of this is strictly necessary, but it makes the proofs shorter. You should interpret anything I say in the strictest way possible.

Some models of $(\infty,1)$-categories work better than others, and so our strictness loses some of them. Let me start by talking about the plan for tomorrow. I'll introduce the axiomatic framework in which we work, an ∞-*cosmos*, which has ∞-categories as objects.

Today we won't need the axioms of ∞-cosmos. We'll instead develop some of the theory of ∞-categories. Let me say, we'll define, introduce, adjunctions and equivalences between ∞-categories, two very fundamental notions, and also limits and colimits inside an ∞-category. I'll prove that these notions relate in the expected way, I'll, for example, prove that right adjoints preserve limits, one of my favorite theorems in category theory. We'll do this working in a strict 2-category with objects ∞-categories, with morphisms ∞-functors, and 2-morphisms the ∞-natural transformations.

Before I dive in, I wanted to say one word about the justification, I want to say a word about where does this 2-category come from? People sometimes get skeptical when we get into the weeds. The first hint that there would be a 2-category like this is a theorem due to Joyal, Rezk, Bergner–Pellisier, also to Verity–Lurie

Theorem 1.1. *There are Cartesian closed model categories whose fibrant objects describe $(\infty,1)$-categories: quasi-categories, Rezk spaces (complete Segal spaces), Segal categories, and complicial sets, i.e., natural marked quasi-categories.*

I want to say next time the following.

Corollary 1.1. *For each of these models, each of these ∞-cosmoi $\mathcal{K}$, there exists a 2-category $h\mathcal{K}$ as described:*

- *the objects are ∞-categories,*
- *the morphisms are ∞-functors, and*
- *the ∞-natural transformations are (defined to be) the 2-cells that arise here.*

So today an ∞-category and ∞-functor should be taken to be the objects and morphisms in one of these four models.

Each of these things is simplicially enriched. The 0-simplices are the ∞-functors, and the homotopy classes of 1-cells are the ∞-natural transformations.

Now let me explain what I mean by a 2-category. We'll have objects, which are capital letters close to the beginning of the alphabet. The morphisms look like $A \xrightarrow{f} B$. I could say $f \overset{\alpha}{\Rightarrow} g$ for 2-cells. Moreover, I should have said the 2-categories have a terminal object 1 and are Cartesian closed in the 2-categorical sense.

First I'll say I have products.

Then if I have a functor $A \times B \xrightarrow{f} C$, that transposes to $B \xrightarrow{f} C^A$ and also $A \xrightarrow{f} C^B$, but also there's a 2-dimensional axiom, that if I have $f \overset{\alpha}{\Rightarrow} g$ in the first case I get the same 2-morphisms under transposition.

Let me explain what a 2-category looks like. I have an underlying 1-category. The thing that is nice, the ∞-natural transformations, I'm saying that I can compose these in a bunch of ways that are always well-defined. They compose *vertically.* So if $f \overset{\alpha}{\Rightarrow} g$ and $g \overset{\beta}{\Rightarrow} h$ then I get $f \overset{\beta\cdot\alpha}{\Rightarrow} h$.

I also have *horizontal* composition, if $f \overset{\alpha}{\Rightarrow} g$ and $h \overset{\beta}{\Rightarrow} k$ where the target objects of f and g are the source objects of h and k, then I get $hf \overset{\beta * \alpha}{\Rightarrow} kg$.

One special case is *whiskering* where one of these is an identity, so I get

$$\begin{tikzcd} X \arrow[r, "a"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "b"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow b\alpha a"] & Y \end{tikzcd}$$

Then more generally, I can compose by pasting, and there is middle four interchange which shows that whenever I draw a directed diagram with natural transformations on each cell it will give a well-defined 2-cell by composition.

1.1. Adjunctions and equivalences.

Definition 1.1. An *adjunction* is comprised of

- ∞-categories A and B,
- ∞-functors $A \xrightarrow{u} B$ and $B \xrightarrow{f} A$, and
- ∞ natural transformations $\mathrm{id}_B \overset{\eta}{\Rightarrow} uf$ and $fu \overset{\epsilon}{\Rightarrow} \mathrm{id}_A$,

So that the "triangle equalities are satisfied", which is

$$\begin{tikzcd} & ufu \arrow[dr, Rightarrow, "u\epsilon"] & \\ u \arrow[ur, Rightarrow, "\eta u"] \arrow[rr, equal] & & u \end{tikzcd} \quad \begin{tikzcd} & fuf \arrow[dr, Rightarrow, "\epsilon f"] & \\ f \arrow[ur, Rightarrow, "f\eta"] \arrow[rr, equal] & & f \end{tikzcd}$$

You can also draw these as pasting diagrams [pictures]

The upshot, the advantage of having a 2-categorical definition, is that all 2-categorical theorems about adjunctions become theorems about ∞-categories. So once we know that this is a good definition for ∞-categories, we get them for free. And if you didn't know them before you can prove them easily.

For example,

Proposition 1.1. (1) *If $f \dashv u$ and $f' \dashv u$ then $f \cong f'$.*
(2) *If $f \dashv u$ and $f \cong f'$ then $f' \dashv u$.*

Proof. For the first case, if we write down the pasting diagram $(\epsilon' f) \cdot (f'\eta)$ this is an inverse isomorphism to $(\epsilon f') \cdot (f\eta')$.

For the second statement, you define η' as $(u\alpha) \cdot \eta$ and define ϵ' as $\epsilon \cdot (f'u)$. □

Here are some other examples.

Proposition 1.2. *Adjunctions compose, so that if $f \dashv u$ and $f' \dashv u'$ and these are horizontally composable, then $ff' \dashv u'u$.*

The proof is an exercise, where you build the unit and counit by hand from the unit and counit of the given adjunctions.

Proposition 1.3. *Given $f \dashv u$ an adjunction between A and B, then $f_* \dashv u_*$ between A^X and B^X ad $u^* \dashv f^*$ (note the reversal) between C^A and C^B.*

The proof is that 2-functors preserve adjunctions, and the 2-functor in question is $(-)^X$ from $h\mathcal{K}$ to $h\mathcal{K}$.

Let me put equivalences in this story.

Definition 1.2. An *equivalence* consists of

- ∞-categories A and B,
- ∞-functors $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, and
- ∞ natural isomorphisms $\mathrm{id}_A \overset{\alpha}{\Rightarrow} gf$ and $fg \overset{\beta}{\Rightarrow} fg$

Then the proposition is that any equivalence can be promoted to an *adjoint equivalence* so that $f \dashv g$ and $g \dashv f$.

The proof is an exercise and I should be more precise, you can keep everything except for one of the two natural transformations.

1.2. **Limits and colimits.** How do we look *inside* an ∞-category, since it's one of four different things. This is a description of Bill Lawvere

Definition 1.3. An *element* of an ∞-category A is a functor $1 \xrightarrow{a} A$. A *generalized element* is $X \xrightarrow{a} A$.

The philosophy of generalized elements is what makes this thing a higher dimensional definition. The 1-cells in the generalized case keep track of higher homotopy.

Definition 1.4. A *terminal element* in an ∞-category A is a right adjoint to the unique functor from A to the terminal object 1. The data is $1 \xrightarrow{t} A$.

I need to give a unit and a counit. But the counit is no data, and then the unit is a map from id_A to $t!$ and it should satisfy that $\eta t = \mathrm{id}_t$.

What's nice about this definition is that it's easy to prove a number of theorems. Let me list one of them.

Proposition 1.4.

- *Right adjoints (and equivalences) preserve terminal elements.*
- *If $A \cong B$ are equivalent ∞-categories then A has a terminal element if and only if B does. When you change models and change back then you might come back in an equivalent ∞-category.*

Proof. If I have $! \dashv t$ and$f \dashv u$ horizontally composable, then $! \dashv ut$ by horizontal composition. For equivalences this is true because we can upgrade the equivalence to an adjoint equivalence. The second result now follows. □

Definition 1.5. I'll let A^J be the ∞-category of diagrams of shape J in A. Recall that $1 \xrightarrow{d} A^J$ corresponds to $J \xrightarrow{d} A$.

Definition 1.6. We say that ∞-category A has *limits of shape J* if the diagonal constant diagram $A \to A^J$ has a right adjoint lim.

Let me say, the counit here, this might be unfamiliar, the counit gives a natural transformation so that given a diagram d, the counit ϵd is the *limit cone* which is a map from the limit to the diagram d.

This is not sufficiently general because sometimes you don't have all of these limits.

Proposition 1.5. *A 2-cell ϵ from fu to id_A is the counit of an adjunction if and only if it is an* absolute right lifting diagram*.*

This means that given any ∞-natural transformation of the following form

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"] \arrow[dr, phantom, "\Downarrow \chi"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

there is a unique cell τ which goes in the upper cell here with ϵ in the lower cell which is equal to χ

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"] & B \arrow[d, "f"] \\ A \arrow[r, equal] \arrow[ur, "u"] & A \end{tikzcd}$$

Lemma 1.1.
- *If ρ is an absolute right lifting diagram, then ρc is for any morphism c.*
- *A vertical composition with the same source object is an absolute right lifting if and only if the first 2-cell is.*

Absolute refers to this first stability under restriction.

Definition 1.7. A diagram $J \xrightarrow{d} A$ has a limit if and only if there exists an absolute right lifting

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"] \arrow[ur, dashed, "\ell"] & A^J \end{tikzcd}$$

with 2-cell λ

Then the theorem which luckily will take no time is that

Theorem 1.2. *Right adjoints preserve limits, so*

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"] \arrow[ur, "\ell"] & A^J \arrow[r, "u^J"] & B^J \end{tikzcd}$$

is an absolute right lifting diagram

Proof. If I take the desired absolute right lifting diagram, then I can vertically compose this with the whiskered counit, and then use the second statement of the lemma. But by 2-naturality of Δ, the counit can slide through the diagonal,

$$\begin{tikzcd}
 & & B \arrow[d, "\Delta"] & & & & B \arrow[d, "f"] \\
 & A \arrow[ur, "u"] \arrow[d, "\Delta"] & B^J \arrow[d, "f^J"] & = & & A \arrow[ur, "u"] \arrow[r, equal] & A \arrow[d, "\Delta"] \\
1 \arrow[ur, "\ell"] \arrow[r, "d"] & A^J \arrow[ur, "u^J"] \arrow[r, equal] & A^J & & 1 \arrow[ur, "\ell"] \arrow[r, "d"] & A^J \arrow[r, equal] & A^J
\end{tikzcd}$$

(with λ and ϵ in the appropriate cells) □

2. Thomas Nikolaus, Higher Categories and algebraic K-theory I

I'll use infinity categories, and we'll learn in Emily's lectures what ∞-categories are and how to use them.

So let's start easily and talk about K_0 as a group completion. So R will be a ring, with unit, not necessarily commutative. Then Proj_R is the category of finitely generated projective R-modules. By $\pi_0(\mathrm{Proj}_R)$ I'll mean isomorphism classes of objects in there. Since there's a symmetric monoidal structure $\oplus$ on Proj_R, I can take two projective modules and take their direct sum, this is a symmetric monoidal category. Then $\pi_0(\mathrm{Proj}_R)$ is a commutative monoid. With that in mind, one defines $K_0(R)$ to be the group completion of this monoid $(\pi_0 \mathrm{Proj}_R)^{\mathrm{gp}}$, the group completion which formally adds inverses.

The basic examples of this invariant that probably everyone has seen, if R is a PID, then $\pi_0 \mathrm{Proj}_R$ is $\mathbb{N}$ so $K_0(R)$ is $\mathbb{Z}$. You could also ask what is $\pi_0 \mathrm{Proj}_{R[x_0,\ldots,x_n]}$ is, and by Quillen–Suslin this is also $\mathbb{N}$ so K_0 is $\mathbb{Z}$.

Another example is $\mathbb{C}[C_2]$, then any projective representation is a sum of trivial and sign representations. Then $K_0(\mathbb{C}[C_2])$ is $\mathbb{Z}\oplus\mathbb{Z}$. You get a similar statement for the group ring of any finite group.

So one thing we could have noted is that this only used the symmetric monoidal structure of the category of finitely generate projective modules.

Definition 2.1. So let $\mathbb{C}$ be a symmetric monoidal ∞-category. Then $\pi_0(\mathbb{C})$ is a commutative monoid and $K_0(\mathcal{C})$ is the group completion $(\pi_0\mathbb{C})^{\mathrm{gp}}$

Let's take examples. If $\mathbb{C}$ is the category of finite sets, then this is symmetric monoidal with disjoint union. This is the coproduct of finite sets. So what is K_0 of that, what are isomorphism classes of finite sets? This is natural numbers, so K_0 is again the integers.

Another example is given as follows. If R is a connective ring spectrum, and I said I'd assume everything about homotopy theory, so just think of rings if you're not familiar with them. This is somehow one of the points of higher category theory, to work with ring spectra like rings. A module M over R is called *projective* if M is a summand in R^n for some finite n. I'm not allowing any shifts here. Then you get a category of projective R-modules just as you did for rings. Then $K_0(R)$ is $K_0(\mathrm{Proj}_R)$. This thing is a 1-category which is the thing I started with if I started there. I'll implicitly consider 1-categories as ∞-categories.

Then it turns out that this is isomorphic to $K_0(\mathrm{Proj}_{\pi_0 R})$, so I'm saying this is $K_0(\pi_0 R)$. Why is this? Because, you can, this is a priori an ∞ category, and you can take the homotopy category $\mathrm{Ho}(\mathrm{Proj}_R)$, and this is equivalent to $\mathrm{Proj}_{\pi_0 R}$.

It's obvious how if you start with a projective module in $\mathrm{Proj}_{\pi_0 R}$ how you can lift. That's not a proof but an indication.

In general, of course, we have that K_0 of any category will just depend on K_0 of the homotopy category. We take iso classes of objects and group complete.

Why is this relevant? A lot of people in the audience will have answers. One of the motivations is that Waldhausen considered Waldhausen A-theory. Wall considered some finiteness obstructions that were eventually K-theoretic. So what's A-theory? If X is a pointed connected space, then there is a ring spectrum which I denote as $\mathbb{S}[\Omega X]$, so ΩX is a based loop space, this is a group object in the ∞-sense in the category of spaces. You can work with simplicial sets and write down a strict model if you want, this is $\Sigma^\infty_+ \Omega X$, this is written $\mathbb{S}[\Omega X]$ because this is exactly analagous to the group ring. So then I define $A_0(X)$ as $K_0(\mathbb{S}[\Omega X])$.

So what is π_0 of this ring spectrum? For a suspension spectrum it's the group ring, so this is $K_0(\mathbb{Z}[\pi_1 X])$.

This is *group completion* or *additive* K-theory. If we're talking about a non-affine scheme, we have to do a different thing, which will be covered in the third lecture.

Are there any questions about K_0?

While this is very classical, this was already discussed by Grothiendieck and others, it was a little tricky to define higher K-groups as a group completion. That's actually where ∞-categories really help. I'll start with a symmetric monoidal ∞-category. So the first thing was to pass to isomorphism classes; the right generalization is to pass to the space of objects

$$(\mathcal{C}, \oplus) \rightsquigarrow \mathcal{C}^{\cong}$$

which is an $_\infty$-monoid in the category of spaces, and I'll group complete that. That will be the definition.

Let me introduce some of the terms I'll use to talk about that. I'll use Segal's definition.

Definition 2.2. An $_\infty$-monoid is a functor from finite pointed sets into spaces

$$M : \mathrm{Fin}_* \to S,$$

so the composite is not equal but there is a 2-cell witnessing it, and so on. You can model this strictly but I don't want to—I want to impose the condition

$$M(\langle n\rangle) \xrightarrow{(\rho^i)_{i\geq 1}} \prod_n M\langle 1\rangle$$

are equivalences.

Here $\langle n\rangle = \{0, \ldots, n\}$ and $\rho^i : \langle n\rangle 1\langle$ is 1 if n is i and 0 otherwise.

I'll take commutative monoids, Mon_∞ to be the full category of functors from Fin_* to S on $_\infty$-monoids.

What are examples? One abuse I'm going to do, if I have a functor, I'll identify it with the underlying object which is $M\langle 1\rangle$.

If A is an ordinary commutative monoid, then this, you can take S a set to $A^S/A^{\{*\}}$, a discrete space.

If M is an $_\infty$-monoid, then the homotopy groups $\pi_* M$ are commutative monoids, but by an Eckmann–Hilton argument, they are groups for $* \geq 1$. They already have one group structure, and then the two structures coincide. So there's precisely one homotopy group which is not a group, which is π_0.

An $_\infty$-group is an $_\infty$-monoid such that $\pi_0 M$ is a group. We have a full subcategory Grp_∞ of Mon_∞. Now what we need to know to finish our definitios is what K-theory is. That's the following theorem. Morally we've just made precise what it means to be a homotopy coherent monoid. The main theorem is that the inclusion from $_\infty$-groups into $_\infty$-monoids admits a left adjoint called *group completion.* For every M an $_\infty$-monoid, there is an initial morphism $M \to M^{\mathrm{gp}}$ with target a group. This is the way one would say group completion in the ordinary land. You can exactly say the same in the category of spaces. Historically this would be in quotation marks and then people would provide models, like the loop space of the bar construction. I'd never write a construction in ordinary land. To prove the theorem, you don't need to give any models, this comes from Bousfield localization of model categories, you can prove this exists without constructing it explicitly.

Then the point is to get as much out of K-theory as I can without any models. Let me also add that there is an equivalence of ∞-categories between Gp_∞ and the infinity category of connective spectra. This is delooping, an ∞-loop machine, pioneered by Boardman and Vogt among others.

Good, so what now? Informally, this means you can do things you do in ordinary algebra in this language. People who know classic homotopy theory will not learn anything new from this.

I'll repeat the definition of K-theory now.

Definition 2.3.
(1) So $K(\mathcal{C})$ is now the spectrum associated with $(\mathcal{C}^{\cong})^{\mathrm{gp}}$. If I ever need the underlying space I'll say $\Omega^\infty K(\mathcal{C})$.
(2) $K(R)$ will be $K(\mathrm{Proj}_R)$, as before
(3) $K_*(\mathcal{C})$ is defined as $\pi_0(K\mathcal{C})$ and similarly $K_*(R)$ is defined as $\pi_*(KR)$.

So now we have two definitions of K_0.

Lemma 2.1.
(1) *The two definitions of K_0 agree.*
(2) *The K group $K_{n+1}(R)$ depends only on the n truncation: $K_{n+1}(R) \cong K_{n+1}(\tau_{\le n} R)$.*
(3) $K(\mathcal{C} \times \mathcal{D}) \cong K\mathcal{C} \oplus K\mathcal{D}$.

So for the first one, I gave you two definitions, one about the group completion of the isomorphism classes, very concrete, and that agrees with the more abstract definition.

Proof. For the first case, we claim that for an $_\infty$-monoid, then $\pi_0(M^{\mathrm{gp}})$ is the same as $\pi_0(M)^{\mathrm{gp}}$. This immediately implies the first one, one is how I defined one and the other the other. So how do we show this? We have a square to show commutes.

$$\begin{tikzcd}
\mathrm{Mon}_\infty \arrow[r, "(\ )^{\mathrm{gp}}"] \arrow[d, "\pi_0"] & \mathrm{Gp}_\infty \arrow[d, "\pi_0"] \\
C\,\mathrm{Mon}(\mathrm{Set}) \arrow[r, "(\ )^{\mathrm{gp}}"] & C\,\mathrm{Gp}(\mathrm{Set})
\end{tikzcd}$$

But Emily told us how to do this, these are all left adjoints so we can pass to the adjoint diagram, and it's obvious that the adjoint diagram commutes, therefore this dagram commutes. This is a little sloppy, we should say that the canonical equivalence of the right adjoints gives a canonical equivalence of the left adjoints.

For the second thing, the same argument shows that the group completion of $\tau_{\le n} M$ is equivalent to the truncation of the group completion $\tau_{\le n}(M^{\mathrm{gp}})$. Then the

statement in question follows from, start with Proj_R, truncate the whole category, so
$tau_{\leq n+1}\mathrm{Proj}_R$ is the same as $\mathrm{Proj}_{\tau_{\leq n}R}$. You might not get the details but I am trying to show you that anyway this is very formal. □

Let me not give a proof of the final property, which is that the product and the coproduct coincide. This is entirely formal, then, from knowing that group completion is a left adjoint.

Why would I care? This you can use to make some simple observations. If you make $A(X)$, then you can define the Waldhausen A-theory spectrum $K(\mathbb{S}[\Omega X])$ so $A_1(X)$ is just $K_1(\mathbb{Z}[\pi_1(X)])$. I won't give an account of A-theory here, but this is how it compares to classical invariants in algebraic topology and it was enlightening for me back in the day.

So what can we say about $K(\mathbb{C}[C_2])$ now? As a *category* it splits into a product, this is $K((\mathrm{Proj}_{\mathbb{C}} 1)\times(\mathrm{Proj}_{\mathbb{C}}\times\sigma))$ so this is $K(\mathbb{C})\oplus K(\mathbb{C})$. This is virtually useless unless we can say something about $K(\mathbb{C})$ which I'll do next time.

So I'll finish with an example of an $_\infty$-monoid for every space X. So you can make a free commutative monoid on any set. So for every space X, you can do something similar, there is a free $_\infty$-monoid on X, with underlying space given as follows. The existence can be deduced formally, but the thing I'll say now is still formal but one level less formal than the other formal thing.

$$\mathrm{Free}_\infty(X)\cong\coprod_{n\geq 0}(X^{\times n})_{h\Sigma_n}.$$

Again I'm not going to justify this. One way to prove this is to compare this to the category of algebras over an operad and write down what this gives.

Next time I'll give you tools to identify these spectra. We can identify a certain spectrum, this is the Barrat–Priddy–Quillen theorem.

Theorem 2.1. *The K-theory spectrum of finite pointed sets is the sphere spectrum, so $K_*(\mathrm{Fin}_*)=\pi_*(\mathbb{S})$.*

So what's the proof?

Proof. So we started with Fin and throw away non-invertible morphisms, but what is this as a space? This is

$$\mathrm{Fin}^{\cong}\cong\coprod_{n\geq 0}B\Sigma_n$$

which is just $\mathrm{Free}_\infty(\mathrm{pt})$. It's not hard once you have the free formula, you can give a map from the free one easily and then you can check whether it's an equivalence on underlying spaces. So this is the same as mapping out of a point. But then the associated spectrum is the free spectrum on a point, which is the sphere. Thus the mapping space $\mathrm{Map}_{\mathrm{Sp}}(K(\mathrm{Fin}),X)$ with X a (connective) spectrum is given by $\Omega^\infty X$. I compute the mapping space in spectra using infinity groups. The underlying space of the E_∞-group of X is the infinite loops of X. But the same is true for the sphere, so $K(\mathrm{Fin})$ is equivalent to $\mathbb{S}$. You get a map one way for free and then check its equivalence on underlying things. □

So I shifted the non-triviality of the theorem into building the machine. Once you assume you know everything about ∞-categories, you get these statements formally. Next time I'll give you tools to help you learn something about the K-theory groups, like if you input a ring. The crucial ideas are due to Quillen, and the idea is to get a handle on the homology of the K-theory spectrum.

3. Tobias Dyckerhoff: Higher Segal spaces I

I'd like to give some introduction in a series of two talks. I'll provide an introduction, these are certain higher categorical structures. There is a notion of a d-Segal space, there are in fact two versions. In the first lecture I'll discuss $d \leq 2$ and tomorrow I'll discuss $d > 2$. Since there are various people in the audience who contributed to this, I'll focus on some of these things so that you can have some discussions with those people. So Thomas already mentioned Segal spaces, they will appear in a different way here.

A *Segal space* is a simplicial space $X_.$ satisfying this condition, for every $n \geq 1$, the n-simplices X_n mapping to the iterated fiber product $X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$, and I'll thinki of this as the maximal edge path in the n-simplex, this map is obtained by restricting along this maximal edge path, and these agree over X_0 because the edges are attached along the vertices.

You can either take this fiber product in the ∞-category of spaces, or take homotopy fiber products, or take a Reedy fibrant replacement of the diagram, and the condition is that this map X_n to the iterated fiber product is a homotopy equivalence.

In Thomas' lecture, this enhanced to finite pointed sets and the 0-simplices were contractible. In this case I lose the additional symmetry from permuting the finite sets, and this actually produces a model for an associative monoid. This provides an explicit combinatorial model for [unintelligible], and this is where this first arose. I'd rather move in a slightly different path.

Rezk used this to give a model for an $(\infty, 1)$-category. One basic such example, take $\mathcal{C}$ to be a small category, then I can associate to it its nerve, which is a discrete example of a Segal space. In fact it's easy to see that all discrete Segal spaces arise as nerves of small categories (and functors among them). Then the Segal condition was an isomorphism, and by allowing it to be a homotopy equivalence I get something associative only up to homotopy, but these are coherent by the higher conditions.

So now I'm supposed to think of this as having a coherently associative composition law. Vice versa, if I'm given such a Segal space $X_.$, I can extract from it a categorical structure, where ob $\mathcal{C}$ is the 0-simplices. Then the morphisms between x and y form a topological space, which is $\{x\} \times_{X_0} X_1 \times_{X_0} \{y\}$. Then the question is how to compose morphisms.

Given a morphism $x \xrightarrow{f} y$ and a morphism $y \xrightarrow{g} z$, I can treat $\{(f,g)\}$ as a point in the space $\{x\} \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \{z\}$ and from the Segal condition it follows that the map from $\{x\} \times_{X_0} X_2 \times_{X_0} \{z\}$ is an equivalence, and so I can pick anything in the (contractible) fiber as a composition:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \{x\} \times_{X_0} X_2 \times_{X_0} \{z\} \arrow[d] \\
\{(f,g)\} \arrow[r] & \{x\} \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \{z\}.
\end{tikzcd}$$

And if I look in X_3 I see that this is associative up to a certain homotopy, which is coherent up to higher homotopies, and we extract an $(\infty,1)$-category.

And there's some fudging where we should have some so-called completeness condition, and then complete Segal spaces are a model for $(\infty,1)$-categories that arose in Emily's talk. Clark Barwick gave a higher model in this flavor using multi-simplicial objects for (∞,n)-categories, which is *not* what we're doing.

I'm going to come straight to the point. I wanted to clear up this misconception—this is not what we're trying to do.

What, then, is a 2-Segal space?

A 2-*Segal space* is a simplicial space $X_{.}$ such that for every $n \geq 2$, and every triangulation of a convex plane polygon—what I mean by that is the following. Label the vertices of your polygon linearly, compatible with the blackboard orientation. I mean a triangulation only involving the vertices on the boundary. Choose any one that you like, and I want to think of this as the 2-dimensional analog of the maximal edge path. So I'll select these triangles, giving a 2-dimensional membrane instead of a 1-dimensional path. I get a map from X_n to a homotopy limit, over all the simplices appearing in this triangulation. Call the triangulation $\mathcal{T}$, and then this is

$$X_n \to \operatorname{holim}_{\Delta^k \subset \mathcal{T}} X_k$$

and the condition is that the map is a homotopy equivalence.

We came up with this definition which was from studying an example. This was in work with Kapranov, but was studied in a different context by Galvez-Corrillo–Kock–Tonks. This is a justification that this is a natural thing to study.

Let me, before trying to understand what this is supposed to be, let me give the first example of such a thing.

Any Segal space is a 2-Segal space. Let me give a sketchy proof. I start with any triangulation of a pentagon, and I'm supposed to think of this in Δ^4, and I think of a map from X_4 to (in this case, [picture]) $X_2 \times_{X_1} X_2 \times_{X_1} X_2$. So I'll apply the ordinary Segal condition, and by the 1-Segal condition, this is $X_2 \times_{X_0} X_2$, removing the maximal edge, and eventually I get to the maximal edge path, and then the whole inclusion into the 4-simplex is supposed to be an equivalence. Then this works by 2-out-of-3 and this generalizes if you are a little more systematic. So this is some weakening of a Segal space in some sense.

You go back to the composition for Segal spaces, and in my pullback something goes wrong. You have no equivalence from $\{x\} \times_{X_0} X_2 \times_{X_0} \{z\}$ to $\{x\} \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} z$ so you don't know whether you have a unique composition, it could be empty, it could have higher homotopy, et cetera.

But if I look at the two possible ways to triangulate a square, I get two different maps from X_3 into $X_2 \times_{X_1} X_2$, and these are weak equivalences. There is still some sort of associativity. Even though there is not a unique composition, even though it's multivalued, it's associative in this multivalued sense.

As in Segal, you get the notion of an associative monoid, here you get a multi-valued monoid, which is associative if I investigate all these different choices.

Now you may wonder what this is good for. What this is good for, in nature there are many 2-Segal spaces that match this model and I'd like to give you one such example next. Curiously, just as the original definition appears in K-theory about group completion, and this also appears in algebraic K-theory as the $S_{.}$ construction of Waldhausen.

Let $\mathcal{A}$ be an Abelian category, then the Waldhausen $S_\cdot$ construction of $\mathcal{A}$ is the simplicial space as follows. I have a single $\{0\}$-object, and then you get a classifying space for 1-simplices, and I'll write it in a redundant way, the groupoid of diagrams of the form

$$\begin{tikzcd} 0 \arrow[r, tail] & A \arrow[d, two heads] \\ & 0 \end{tikzcd}$$

and the 2-simplices are of the form

$$\begin{tikzcd} 0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d, two heads] & B \arrow[d, two heads] \\ & 0 \arrow[r, tail] & A' \arrow[d, two heads] \\ & & 0 \end{tikzcd}$$

where the squares are biCartesian. and for three it's

$$\begin{tikzcd} 0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, tail] \arrow[d, two heads] & C \arrow[d, two heads] \\ & 0 \arrow[r, tail] & A' \arrow[r, tail] \arrow[d, two heads] & B' \arrow[d, two heads] \\ & & 0 \arrow[r, tail] & A'' \arrow[d, two heads] \\ & & & 0. \end{tikzcd}$$

So Waldhausen says that $K(\mathcal{A})$ is $\Omega|S_\cdot(\mathcal{A})|$ if you are working somewhere were exact sequences don't split. Here I want to mod out by relations coming from short exact sequences as well.

So then the point is that this $S_\cdot$ construction is precisely a 2-Segal space. There is actually this algebraic construction as a monoid. There is an intrinsic algebraic interpretation, captured by this notion of a 2-Segal space.

We gave the same definition and the same theorems totally independently, so this GC–K–T and also D–K.

Theorem 3.1. *The $S_\cdot$-construction is a 2-Segal space.*

Let me give you the sketch of a proof for why this is true.

Proof. The lowest condition is that, say, I have two 2-simplices in the $S_\cdot$ construction glued together along a face in this way:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \arrow[dl, dash] \\ 0 \arrow[r, dash] & 3 \end{tikzcd}$$

and this is supposed to give me a 3-simplex. So this is by filling with the cokernel:

$$\begin{tikzcd}
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, tail] \arrow[d, two heads] & C \arrow[d, two heads] \arrow[dd, bend left] \\
& 0 \arrow[r, tail] & A' \arrow[r, tail] \arrow[d, two heads] & B' \arrow[d, two heads] \\
& & 0 \arrow[r, tail] & A'' \arrow[d, two heads] \\
& & & 0.
\end{tikzcd}$$

Then by the third isomorphism condition, the three cokernels I can take fit in an exact sequence and the bottom square is coCartesian. ☐

This can also be done for exact categories, and even for a set-theoretic context, with a proto-exact structure (exact without linearity) and also for stable infinity categories and relative $S_\cdot$-constructions, where you do this and get a hybridization of the nerve of $\mathcal{B}$ with $S.(\mathcal{A})$ where we do this on $f : \mathcal{A} \to \mathcal{B}$. It doesn't work for Waldhausen categories because there you only have cofibrations and not the duality. You need the condition where in a square, if one parallel pair is monic and the other parallel pair epic, then the square is a pushout if and only if it's a pullback.

So are there other examples? There are some Hochschild-style examples, but I want to present a theorem by several people in the audience.

Theorem 3.2. *(Bergner–Osorno–Ozoncva–Rovelli–Schiembauer)*
There is an equivalence of categories between unital 2*-Segal sets and augmented stable double categories (the maximal possible thing where you can take the $S_\cdot$ construction).*

This is a category object in the category of categories. Stability is designed so that this game works, so that you can complete spans.

We'll see a version of this statement for $d \geq 2$ in the next lecture as well. What *are* these 2-Segal spaces?

Theorem 3.3. *(Walde) There is an equivalence of ∞-categories between unital* 2*-Segal spaces and invertible planar ∞-operads.*

I should say what unital means. If an edge in a 2-simplex is degenerate, it's the same as a 1-simplex, i.e., $X_0 \times_{X_1} X_2 \cong X_1$ and likewise on the other side.

Maybe I'd like to give a sketch of the proof, which is actually very slick.

Proof. Exhibit the simplex actegory Δ as an ∞-categorical localization of the category Ω of planar rooted trees Ω (Moerdijk–Weiss, later Cisinski–Moerdijk). You take a tree, collapse to a corolla, and count your outputs. If you localize along a certain class of collapse maps, then the localization gives you precisely the simplex category. What you learn is that, if you look at simplicial spaces $\Omega \xrightarrow{\pi} \Delta$, then simplicial spaces realize as pullback as invertible dendroidal spaces. Inside the simplicial spaces the unital 2-Segal spaces, then these precisely correspond to complete Segal condition in the language of trees, and so you get the statement of the result. ☐

You can now unravel in this language what it means to have a multivalued composition law. This is one way of making this precise in existing ∞-categorical terms.

We've produced many examples. We know how to rigorously think of them as multivalued composition laws. Now the question is, what can we do with this? What are applications of this notion of a 2-Segal space. I'd just like to survey a few contexts in which you can apply this theory.

The first application, and this was the originally conceived context in which we wanted to study these 2-Segal spaces, is the notion in representation theory of Hall algebras, these come from any Abelian category satisfying certain finiteness conditions. You take an isomorphism class of object and try to multiply it by another, and you want to say this is $[X]$ when $[X]$ appears in an extension of A by B or the other way around. So in one version

$$[A]\cdot[B] = \sum_{[X]} \frac{\#(A \hookrightarrow X \twoheadrightarrow B)}{\#\operatorname{Aut}(A)\cdot\#\operatorname{Aut}(B)}[X]$$

but fro this to make sense these had better be finite sets so usually this is over finite fields. Then we noticed that somehow what you do here, this construction factors through the $S_\cdot$ construction, where this is, remember you had this correspondence

$$X_1 \times_{X_0} X_1 \leftarrow X_2 \rightarrow X_1$$

and you do this by some sort of pull push construction. Then you deduce the associativity of the formula by the kind of Segal condition.

So we wanted something for 2-periodic categories and this construction doesn't work. So 2-Segal spaces were the next best thing to giving associativity without finiteness.

Other audience members have given, Penney gave a categorified Green's theorem. Wald, in his Master's thesis gave Hall monoidal categories. Yong gave a [unintelligible] $S_\cdot$-construction, and Pogutte gave equivariant motivic Hall algebras which came up in this kind of investigation.

There are applications to topological field theories, where you try to understand 2-periodic derived categories, where you don't just look at a simplicial object but a *structured* 2-Segal object, then these, we've constructed a 2-dimensional open topological field theory valued in correspondences, and this is supposed to be an equivalence, this should appear soon in Stern. The various structures encode structures beyond the simplex category, so e.g., if you lift to the cyclic category you get an oriented field theory. If you're less ambitious and give the paracyclic category, you get a framed theory, and so forth. So you get a dictionary relating reductions of structure group to the crossed simplicial categories. You expect a cobordism hypothesis in each of these contexts.

So one striking application, if you do this to a stable ∞-category, you get a framed TFT which associates to a framed punctured surface an ∞-category, which is what is called (a topological version of) the Fukaya category. Kontsevich said that in this context there should be a combinatorial way to construct such a thing. So for example people have been using this framework to show versions of mirror symmetry using this formalism. This is maybe a survey of what there is to say about 2-Segal spaces. In the next lecture I'll talk a little about what to do in higher dimension and how these are supposed to fit together.

4. John Francis: Factorization homology I

Thank you very much for the introduction and the invitation to speak here. It's a pleasure to be about to talk to you about this stuff. I'll speak about factorization homology, as suggested by my title. I'll talk about what it is and what it's supposed to be. It's supposed to be the answer the question of how to study multiplicative invarianst by local-to-global methods, i.e., some sort of sheaf theory. I refer to how this invariant behaves on disjoint unions.

One of the big ideas of 20th century mathematics is how to study additive invariants with sheaf theory. We'd say $Z(X \coprod Y) = Z(X) + Z(Y)$ (if these were numbers) or $Z(X) \oplus Z(Y)$ if this is valued in, say, chain complexes or spectra. So then the meta-theorem you'd like to achieve is that

$$Z(X) = \int_X \mathrm{class}_X$$

or maybe that $Z(X)$ is a homology with coefficients in $Z(*)$.

Some examples of this are Gauss–Binet or Hirzebruch signature theorem. Likewise the Atiyah–Singer index theorem fits into this, the Dirac operator doesn't look local at all, but it's integrating the $\hat{A}$-genus. Whether this can be represented by homology, you need to check not just additivity but excision.

Now we want to thing that $Z(X \coprod Y)$ is $Z(X)Z(Y)$ or $Z(X) \otimes Z(Y)$ in chain complexes (or smash product in spectra).

So here's an idea, it's not, they didn't put it quite this way, but it's embedded in the work of Beilinson and Drinfeld, but it's to use Ran space, the space of finite non-empty subsets. For additive invariants you can use sheaves valued in the space itself. A sheaf on a disjoint union will never have this multiplicative property. Instead, construct some auxiliary space that has this multiplicativity built into it.

Definition 4.1. The configuration space $\mathrm{Conf}_i(X)$ is the space of ordered i-tuples,

$$\mathrm{Conf}_i(X) = \{(x_1, \ldots, x_i) | x_j \neq x_k, j \neq k\} \subset X^i$$

given the subsace topology of the Cartesian product. There's an action of the symmetric group on this and the quotient is the unordered configuration space.

Definition 4.2. For X connected, the Ran space $\mathrm{Ran}(X)$ is the disjoint union of unordered configuration spaces

$$\cup_{i \geq 1} \mathrm{Conf}_i(X)_{\Sigma_i}$$

starting at $i = 1$

So the most natural topology would be the disjoint union but I want the strata to come together as points collide.

There are natural maps $X^i \to \mathrm{Ran}(X)$ which take an i-tuple to the image in X. If they are all in the same point in X, this would go to just the configurations of 1 point.

We have the naturally defined maps such that these maps get the strongest or finest topology, the fewest convergent sequences such that these maps $X^i \to \mathrm{Ran}(X)$ are continuous for all i.

That describes the topology on $\mathrm{Ran}(X)$. Where this arose, this was originally called the infinite symmetric product and was studied by Borsuk and Ulam, and Bott corrected a mistake of Borsuk studying the Ran space of the circle. But none of them had Beilinson and Drinfeld's insight.

The continued definition, the Ran space for X general will require you to have one point in each connected component.

Let's note that this has the property that $\mathrm{Ran}(X \coprod Y) \cong \mathrm{Ran}(X) \times \mathrm{Ran}(Y)$. It's behaving like an exponential space, taking $\coprod$ which is like $+$ to $\times$ which is like $\cdot$. So sheaf theory on the Ran space might be useful for studying multiplicative things.

So what's the recipe for factorization homology? All of this will be what we call the α-version. Everything I say is joint with David Ayala. I sometimes forget to say that because I think everyone knows it. Some parts are joint with Hiro Lee Tanaka and Nick Rozenblyum as well.

So this has two inputs. One is X and is geometric, of dimension n for the rest of today. The other is algebraic and is A. It depends at the very least on n and maybe also on X. Once we have the inputs we can do a construction. This construction produces $\mathcal{F}_A$, a sheaf or maybe a cosheaf, or something like that, on $\mathrm{Ran}(X)$. Then we get factorization homology.

So the factorization homology of X with coefficients in A, will be written with an integral sign (it's some categorical integration)

$$\int_X A = H_*(\mathrm{Ran}(X), \mathcal{F}_A).$$

So that's the recipe. For the rest of today I'll follow this recipe in some interesting examples where the dimension varies, the geometric structure, and the algebraic structure varies and see what we get.

For advertising purposes, let me tell you some advantages.

(1) You can take something you already know and see what you get out of this machinery.
  - You get "$\otimes$-excision"
  - The Ran space is filtered by cardinality and you get a filtration on your homology and associated spectral sequence. Filtrations and spectral sequences are powerful.
  - There is a duality to factorization homology, which dualizes both Poincaré duality form manifolds and Koszul duality for algebras.

(2) If you had something you wanted to understand but didn't know how to define it, this may tells you that you have a definition and that it's well-definition.

[Does this have an adjoint?]

No. At the level of numbers you can exponentiate or take logs. Exponentiation lifts but logarithms don't. If there was a categorical logarithm you could just take it and then study the additive invariants. If it existed I think this wouldn't be an interesting direction to study.

[If you exponentiate an additive invariant?]

That's the easy direction but it's not so compelling to tell someone with the Atiyah–Singer index theorem "that's great, we can do the multiplicative version, you just need to [unintelligible] infinitely many times."

Okay, let's look at dimension one. Then X is a 1-manifold and A is an associative algebra. Then $\mathcal{F}_A$ is a cosheaf on $\mathrm{Ran}(X)$, for specifity let's say $\mathrm{Ran}(\mathbb{R})$. It's good to think in terms of the stalk. The stalk at an unordered i-tuple is $A^{\otimes i}$. Then on each configuration layer you get a constant cosheaf. This will be constructible with respect to that filtration. When you change from a generic fiber to a special fiber, what happens, what's the specialization map?

So if $\Delta^1 \xrightarrow{f_t} \mathrm{Ran}(\mathbb{R})$, whre $*$ goes to (x_t, y_t), then you'll need a map $A^{\otimes 2} \to A$, and this comes from the multiplication.

This used a choice of framing or orientation so you know what to multiply.

Then the factorization homology of $\mathbb{R}$ with coefficients in A is A, and for the circle it's Hochschild homology,

$$\int_{\mathbb{R}} A = A \qquad \int_{S^1} A = HH_*(A).$$

So that was dimension one. What about complex dimension 1. Here I'll be even more brief. This was Beilinson and Drinfeld's original motivation. In the early 90s, they wanted to do conformal field theory in a geometric rather than an algebraic way. Here X is a curve over $\mathbb{C}$, let's say smooth, and A is a vertex or chiral or factorization algebra, these are the sorts of algebraic input which are allowed.

Now one can define this object $\int_X A$, called by Beilinson–Drinfeld the chiral homology. This was a new object, not considered previously, but related in the sense that H_0 of $\int_X A$ were the "conformal blocks", a.k.a. the correlation functions of the CFT associated to A. Given a surface you get a vector space and operators and this calculates the vector space at the zero dimensional level, but then you have the higher homology and that was new.

Now let's do this over a finite field. Let X be a curve over a finite field. So lek A be $C^*(BG)$, where G is an algebraic group, semi-simple, simply connected (at least for the purposes of the theorem I'm about to state). Then there's a construction in real dimension 1.

Theorem 4.1 (Gaitsgory–Lurie)**.** *In this case $\int_X A$ is $C^*(\mathrm{Bun}_G X)$.*

One can perform weighted counts of this using the advantages I mentioned. This used the entire factorization homology package of techniques, using a lot of the work of Beilinson and Drinfeld.

All right, so real dimension n. There will be a bunch of examples here.

- One source of examples come from n-fold loop spaces.
- Another source of examples come from Lie algebras, or rather certain types of enveloping algebras of Lie algebras.
- Another source of examples come from commutative algebras.
- Another source of examples come from shifted Poisson (n-Poisson) algebras.
- All of these are related to one another, and let me give one more, observables, TQFT

All of these will be for X an n-manifold, and let's suppose X is framed, so $T_X \cong \mathbb{R}^n \times X$. This is restrictive but let's just assume it, we might have a reason.

So one example, for B pointed, let $A = \Omega^n B$ or $C_* \Omega^n B$. The claim is that for such a thing we can construct a cosheaf on the Ran space. So consider the category Disk_n with objects disjoint unions of $\mathbb{R}^n$ and morphisms open embeddings. Then there's a functor $\mathrm{Disk}_n \to \mathrm{Spaces} \xrightarrow{C_*} \mathrm{Ch}$ of compactly supported maps to B, that is, $\mathrm{Map}_c(-, B)$. That is, maps from U to B so that there is some compact X in U outside of which the function is constant at the base point.

I can restrict to π_0-surjections, which is equivalent to the exit path category of $\mathrm{Ran}(X)$ (this is Ayala–Francis–Tanaka). So then using Lurie's theorem on constructible sheaves, this gives a functor, let me just say, a cosheaf on $\mathrm{Ran}(X)$ with stalks $C_*(\Omega^n B)^{\otimes i}$.

I just described how this gives a cosheaf on the Ran space of X. If I wanted to stop in spaces and not apply chains, that's also okay, then I use the Cartesian product instead.

So what's factorization homology. There's a theorem called non-Abelian Poincaré duality due separately to Salvatore, Segal, and Lurie. This says

$$\int_X \Omega^n B \cong \mathrm{Map}_c(X, B)$$

if B is n-connective. This is compatible with taking chains.

All right, Lie algebras, here there's a thing $U^{\mathcal{E}_n}(\mathfrak{g})$ called the $\mathcal{E}_n$-enveloping algebra of a Lie algebra, generalizing the usual enveloping algebra for $n = 1$. These are n-disk algebras, and you use a similar procedure. The calculation here, in characteristic zero, this is

$$\int_X U^{\mathcal{E}_n}(\mathfrak{g}) \cong C_*^{\mathrm{Lie}}(\Omega_c^*(X, \mathfrak{g})).$$

For commutative algebras, then you get the tensor $X \boxtimes A$ (ring spectra being tensored over spaces) and this is homotopy invariant.

For Poisson algebras, there is no good answer. So you can characterize this using something like quantization. This is not sufficiently studied so far, if anyone is so inclined.

Let me say, the commutative example is homotopy invariant and the Lie example is proper homotopy invariant, and this is not an accident, it's an artifact of the duality between commutative and Lie, an example of the duality I mentioned.

By the way, no one has taken me up on my offer, which is still open.

[What about in the commutative case]

The commutative case is Spanier–Whitehead dual to the a Lie case, one of them is Sym with a shift of some cohomology and the other is Sym with no shift of a homology.

So TQFT, I don't mean in the sense of Atiyah's axioms, but rather what a physicist means when he or she says a quantum field theory, which has various pieces of structure. You have observables $\mathrm{obs}_{\mathcal{F}}(\mathbb{R}^n)$ and you try to take expectation. These can be extended by zero so form a covariant functor on open sets. So this forms a cosheaf on $\mathrm{Ran}(X)$ where the stalks are $A^{\otimes i}$ over a configuration of i distinct points.

This needs to be processed, but a good reference is this great book of Costello–Gwilliam. Whatever we're doing, there is a map which takes local observables and glues them together to get global observables,

$$\int_X A \to \mathrm{Obs}_{\mathcal{F}}(X).$$

We just constructed the left hand side with some machinery. You can ask whteher it's an equivalence, and it's an equivalence (morally) for $\mathcal{F}$ perturbative, if you deformed an $\mathcal{E}_n$-type structure to get something quantum, in that case, starting with a Poisson with

[What does morally mean?]

I can't tell you what 'morally means'. I guess I mean it isn't true. [laughter] That's not supposed to be a statement about the world. [more laughter]

A sigma-model with affine target, then I can erase 'morally'.

The most interesting results were in dimension 1 over finite fields or even for dimension 1 for $\mathbb{C}$. You could ask for higher dimensions over $\mathbb{C}$. I wrote something with Dennis Gaitsgory about this but it's kind of the wrong thing. Describing the homology of the mapping spaces it was important to have n-connectivity and here we have the importance of the perturbativity.

You may ask whether it's possible to do this without these restrictions, and that's what the β-version is about. The basic idea is to replace the Ran space with the moduli space of stratifications of X. That's what I'll continue with tomorrow.

5. Sarah Yeakel: Isovariant homotopy theory

Thank you for the invitation. Can you hear me? I'll talk about isovariant homotopy theory, I decided to interpret the ttle of the conference sort of broadly. This is higher structure in a different way. No ∞-categories. This is work in progress with Malkiewich, Merling, and Ponto. It's very in-progress, I'm not sure what names to put with what theorems for the instance. First let me tell you what isovariance is.

Let G be a finite group. Then X and Y are spaces with continuous G-action. A map $f : X \to Y$ is *equivariant* if it preserves the G-action, that is, if $g \cdot f(x) = f(g(x))$. Then $G_x \subset G_{f(x)}$. If you have a free element, that means it's stabilized by the trivial element, it can be mapped anywhere in Y. A map is *isovariant* if it is equivariant and $G_x = G_{f(x)}$. So fixed points go to fixed points, free points go to free points, and at every level in between the isotropy groups are preserved.

An example, if we map the point with a trivial action to the disk with a flip action, I can go anywhere on the fixed line, this is equivariant and isovariant. But the map back is not isovariant.

Any equivariant injective map is isovariant but there are interesting maps that are isovariant but not injective.

Why bother? If you start with a continuous map between a space and itself, then you can ask, when can f be homotoped to a fixed-point free map? This is related to the Reidemeister trace, if this vanishes then you can do it, and vice versa.

You could throw in G-actions, which has to do with Nielson numbers, this has also been studied. So you can ask instead, when can f be isotoped to a fixed point free map? That's a homotopy through homeomorphisms. There is not a complete obstruction. The idea is that isovariant homotopy theory may help with this. So $f^n : X^n \to X^n$ is isovariant with respect to the Σ_n action and modifying f by isotopy is modifying n by isovariant homotopy.

There are a couple of papers by Klein–Williams (Homotopical intersection theory I and II), who look at a variant of the fixed point picture, saying if we have a map $M \xrightarrow{f} N$, if we have a map between manifolds and a submanifold $Q \subset N$, we can ask when can f be homotoped off Q? We're asking for a lift up to homotopy

$$\begin{tikzcd} & N \setminus Q \arrow[d] \\ M \arrow[r, "f"] \arrow[ur, dashed] & N \end{tikzcd}$$

So we can ask instead about the fixed point problem, of homotoping off the diagonal

$$\begin{tikzcd} & M \times M \setminus \Delta \\ M \arrow[r, "1\times f"] \arrow[ur, dashed] & M \times M. \end{tikzcd}$$

So in the second paper they asked about this with G-actions, and they recovered the Nielson numbers. So what if we do this isovariantly, do we get our complete obstruction?

So I want to talk about the homotopy theory. So first I want to talk about what isovariant weak equivalences are. I should give a little bit of an argument here, we tried to do this isovariantly and they got out of hand. Maybe if we get model structures on G-spaces with isovariant maps, these sorts of things will just pop out.

I'll focus on cyclic groups $G = C_p$. An isovariant map $f : X \to Y$ induces a map $f^G : X^G \to Y^G$ and also f^c (all the notation is a free-for-all) on the free points $X \smallsetminus X^G \to Y \smallsetminus Y^G$. This restricts to something I'll call f^N, where you subtract off the fixed points $f^N : NX^G \smallsetminus X^G \to NY^G \smallsetminus Y^G$. I'll denote NX as $NX^G \smallsetminus X^G$

Then a G-space is a pushout here. So

$$\begin{tikzcd} NX \arrow[r] \arrow[d] & X \smallsetminus X^G \arrow[d] \\ X^G \arrow[r] & X. \end{tikzcd}$$

and we want to keep track of all of this. So let γ be a path in X with $\gamma(0)$ in the fixed points and $\gamma(0,1]$ outside of the fixed points. Call this a "base path". The nth *neighborhood homotopy group* of X at γ is

$$\pi_n^N(X,\gamma) = \{S^n \times I \to X | * \times I \xrightarrow{\gamma} X, S^n \times \{0\} \mapsto X^G, S^n \times (0,1] \mapsto X \smallsetminus X^G\}$$

up to isovariant homotopy relative to γ.

Let's take the disk with 180 degree rotation. So I have γ starting at my fixed point. One entire end of the cylinder will be my point. This one contracts to γ but this one doesn't [pictures]. So you can see that $\pi_1^N(D,\gamma) \cong \mathbb{Z}$ in this case. We call this neighborhood homotopy because it's telling you about what happens in a deleted neighborhood.

An isovariant weak equivalence is an isovariant map with f^G and f^c weak equivalences and f induces isomorphisms on all neighborhood homotopy groups.

So for example, if I take my 180 degree rotation on the disk, the inclusion of the point into the disk is not an isovariant weak equivalence. There is no neighborhood homotopy group. These do deserve the name weak equivalence, because they satisfy 2 out of 3 and are closed under retracts. An isovariant homotopy equivalence is also an isovariant weak equivalence.

I want to give two model structures here. The cofibrantly generated model structure for equivariant G-spaces, we have cells $G/H \times D^n$ and the attaching maps are along $G/H \times S^{n-1}$. You can think of this G/H as keeping track of what fixed points you land in. If you wanted to build a G-CW-complex that is the disk with a flip, you glue 2 fixed 0-cells, and one fixed 1-cell and one free 1-cell onto it and 1 free 2-cell. We can't build this space isovariantly. If we want the attaching maps to be isovariant, then the boundary is landing in the fixed points, but the cell itself is free. The two-cell, the map is landing in both fixed and free things.

This leads us to the first of the two model structures, which is the elementary model structure. Let I' be the space which is $G \times I/G \times \{0\}$, with $G \times 0$ fixed and $G \times I$ free. This is an elementary cell. The extra cell that we need so that the attaching maps are isovariant—

Theorem 5.1. *There is a cofibrantly generated model structure on $G-\mathrm{Top}^{\mathrm{isvt}}$ with weak equivalences the isovariant weak equivalences and generating cofibrations the equivariant cells and these new ones $S^{n-1} \times I' \to D^n \times I'$.*

Then the generating cofibrations are just what you'd want them to be, what they were for equivariant, and then the extra elementary cells $D^n \times I' \times 0 \to D^n \times I' \times I$.

If I want to build the disk, I need the 2-fixed 0-cells, two free 0-cells, I want to glue in my 2-cell, it will look like $D^1 \times I'$, I'll glue it in along [pictures].

The conjecture is that this generalizes for all finite groups. For example, if we were to take a cyclic group of order p^2, then we have three types of points, and 3 elementary 1-cells, and 1 elementary 2-cell. We can label them by subgroup inclusions so you have $I^{e<p}$ and $I^{p<p^2}$ and $I^{e<p^2}$. You get a two-cell where one corner is fixed by p^2 and the rest of an edge containing it by p and then the rest by e.

A drawback of this model structure, notice we didn't get to glue in this 1-cell for free, so we won't be able to build things like the disk with rotation. Some manifolds are not cofibrant.

Conjecture 5.1 (Malkiewich (laughter))**.** There is a cofibrantly generated model structure with the same weak equivalences and the generating cofibrations $\{G/H \times S^{n-1} \to G/H \times D^n\}$, and we'd be able to construct this [picture] if we had boundary data, the boundary is $S^{n-1} \times I' \cup \partial\partial I' \times D^n \to D^n \times I'$. The boundary here [picture] is the entire figure eight. In this structure manifolds are cofibrant. The generating acyclic cofibrations are pushout products of cofibrations with $0 \to I$. So now you do get that.

A pro is that this gives you more things and manifolds are cofibrant but the cons are that this isn't fully proved and is a little harder.

There's an isovariant homotopical intersection theory. So we have $M \to N$ and we want to lift to $N \smallsetminus Q \to N$ up to homotopy.

So that's the same thing as a section of the pullback. If I look at a fibration $E \to B$, when is there a section, and they come up with an obstruction to a section. We can look at the unreduced suspension $S_B E$, where we glue $E \times I$ to B on the ends, and if S_+ and S_- are homotopic over B, then that is the obstruction. Let me state this more carefully.

If a section exists, $B \to E$, then S_+ and S_- are homotopic over B. If S_+ and S_- are homotopic over B and some connectivity and dimension requirements, then a section exists.

Thy attribute this to [unintelligible]. In the second paper they do this equivariantly. To see this, one direction is easy, and the other direction a little harder. So for equivariant, you need some further connectivity and dimension requirements. What we've shown is the same thing isovariantly, except we have to throw in some more conditions, we need e.g., that the neighborhood for E mapping down to the fixed points is a Hurewicz fibration, then an isovariant section exists.

Theorem 5.2. *If S_+ and S_- are isovariantly homotopic over B plus extra conditions, then an isovariant section exists.*

This can then be converted, when will these conditions pull back? You need to throw in a bunch of other conditions. Then what happens when we look at the graph question, looking at $M \to M \times M$, can it lift away from the diagonal. We're not guaranteed that this is the identity on the first coordinate, and so we need to project and see that we have a lift in the projected diagram. This is a little messy because it's not an isovariant map.

6. July 3: Emily Riehl: The model-independent theory of $(\infty, 1)$-categories II

[missed the first couple of minutes]

Before defining infinity cosmos, I want to motivate a little bit by talking about a theorem I mentioned yesterday

Theorem 6.1 (Joyal, Rezk, Bergner/Pellisier, Verity/Lurie, Joyal–Tierney)**.** *Right Quillen equivelences of Cartesian closed model categories whose fibrant objects are $(\infty, 1)$-categories.*

The left adjoint in each of these cases preserves products.

As a corollary, each model category is enriched over simplicial sets (with the quasi-category model structure) and the right adjoints are simplicial.

I'd interpret an equivalence of homotopy theories as an equivalence of $(\infty, 1)$-categories. A model category can be promoted up to equivalence to a simplicial model category (a different enrichment, over the model category whose fibrants are Kan complexes). This is a categorification more familiar in homotopy theory, we're enriched over a model of $(\infty, 1)$-categories, not over a model of ∞-groupoids. So this is an equivalence of $(\infty, 2)$-categories, not just of $(\infty, 1)$-categories.

I realize that is a bit confusing, any questions?

We're not going to use model categories in any way. It's familiar to some people and it gives some motivation for what I'm about to do.

The idea is that an infinity-cosmos axiomatizes the fibrant objects together with the fibrations between them and the weak equivalences between them in a category like this, a model category enriched as such over the model structure on quasi-categories.

We've thought about these axioms in two different forms. They're a bit simpler if we assume that the fibrant objects are also cofibrant. It makes everything just a little bit easier. The axioms are properties that are obviously true in examples like that. Many but not all of our examples come from situations like this (more on this later). One more important remark: we recast ∞-*category* and ∞-*functor* as technical terms to mean the objects and morphisms in an ∞-cosmos. It's going to be similar to the common usage in the literature, if you mean quasi-category or complete Segal space or (∞, n)-category or (∞, ∞)-categories, those are all ∞-categories.

Definition 6.1. So, an ∞-*cosmos* is a category $\mathcal{K}$ (whose objects are ∞-*categories* and morphisms are ∞-*functors*) along with a specified class of maps called *isofibrations* and depicted $\twoheadrightarrow$.

There are three axioms.

- **enrichment:** The category should be enriched over simplicial sets with the homs all quasi-categories, I'll write $\mathrm{Fun}(A, B)$ for the quasicategory of maps from A to B.
- **limits:** The category $\mathcal{K}$ should have terminal objects, products, pullbacks of isofibrations, limits of towers of isofibrations, and simplicial cotensors. All of the universal properties are enriched strictly over simplicial sets (I'll spell

this out for the cotensor). If I have ∞-categories A and X and a simplicial set U, then I get an ∞-category A^U in $\mathcal{K}$ so that there is an isomorphism $\mathrm{Fun}(X, A^U) \cong \mathrm{Fun}(X, A)^U$. I do mean isomorphism, not something weaker. Part of the philosophy is working as strictly as possible to simplify proofs.

isofibrations: They have the closure properties you expect and maybe some you didn't. They contain all isomorphisms, all terminal morphisms, are closed under composition, stable under pullback or forming the limit of a tower of isofibrations, and something called Leibniz cotensors, so if I have a monomorphism $U \hookrightarrow V$ of simplicial sets and an isofibration $E \twoheadrightarrow B$ then there is a map $E^V \to E^U \times_{B^U} B^V$ and that should be an isofibration. Finally, for all X and any isofibration $E \twoheadrightarrow B$, the postcomposition map $\mathrm{Fun}(X, E) \to \mathrm{Fun}(X, B)$ is an isofibration in quasicategories.

What ∞-cosmology is, is more a philosophy and less a definition. If you need more axioms you should add them and if you don't need them all, then you should get rid of some.

Definition 6.2. A map in an ∞-cosmos $\mathcal{K}$ is an equivalence $\xrightarrow{\sim}$ if and only if when I look at the postcomposition $\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ I get an equivalence of quasi-categories. So equivalences are defined representably in quasi-categories. It's a trivial fibration $\overset{\sim}{\twoheadrightarrow}$ if it's an equivalence and an isofibration.

So the underlying 1-category is a category of fibrant objects in the sense of Brown. All of these can be proven from the axioms I've put here. One of these lemmas says that if I have a functor $A \xrightarrow{f} B$ of ∞-categories, I can factor it as an equivalence $A \xrightarrow{\sim} Pf$ which is a section of an trivial fibration, followed by an isofibration $Pf \twoheadrightarrow B$. So these are not homotopically meaningful. They play a couple of roles, one is that they can let us replace a homotopy commutative diagram with a strictly commutative diagram.

Let me give some examples. QCat is an ∞-cosmos, so these are quasi-categories, simplicial sets so that inner horns have fillers. The isofibrations are an inner horn lifting condition along with an isofibration lifting condition

$$\begin{tikzcd} \Lambda^{[unintelligible]}[[unintelligible]] \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta[[unintelligible]] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The equivalences are the maps $A \xrightarrow{f} B$ such that there exists a map $B \xrightarrow{g} A$ along with some homotopy condition involving maps $A \to A^{\mathbb{I}}$ and $B \to B^{\mathbb{I}}$.

Other examples include complete Segal spaces, Segal categories, 1-complicial [unintelligible], the n-versions of these, complicial sets, categories, slices $\mathcal{K}/B$ and $\mathcal{K}^B$ (In particular $\mathcal{K}_*$).

Now how did I get to the 2-category I worked in last time? I should review the homotopy category of a quasi-category.

Proposition 6.1. *The nerve has a left adjoint.*

I agree with Thomas that We should just identify a strict 1-category with its image in quasi-categories. This adjoint is ho and $\mathrm{ho}\, A$ will have its objects the vertices of A and morphisms the 1-simplices up to homotopy.

So f is equivalent to g if there is a 2-simplex filling

$$\begin{tikzcd} & b \arrow[dr, equal] & \\ a \arrow[ur, "f"] \arrow[rr, "g"] & & b \end{tikzcd}$$

Now ho preserves products and by applying ho to morphism sets I get an adjunction between Cat-enriched categories and QCat-enriched categories, where I take $\mathcal{K}$ to $h\mathcal{K}$. So

Definition 6.3. The *homotopy* 2*-category* of an ∞-cosmos $\mathcal{K}$ has the same objects and hom categories $h\operatorname{Fun}(A,B) \coloneqq \operatorname{ho}(\operatorname{Fun}(A,B))$.

So the objects are the ∞-categories, the objects of $\mathcal{K}$, the morphisms are the ∞-functors in $\operatorname{Fun}(A,B)$. The ∞-natural transformations are one-cells in $\operatorname{Fun}(A,B)$ up to homotopy.

This is like the 2-category version of the homotopy category of a model category.

I want to prove something easy and then something harder about the homotopy 2-category. In the first talk we were talking about things that are Cartesian closed, so I want to talk about where that comes from.

Definition 6.4. An ∞-cosmos $\mathcal{K}$ is Cartesian closed if $\operatorname{Fun}(A\times B,C)\cong \operatorname{Fun}(A,C^B)$ and $(-)^A : \mathcal{K}\to\mathcal{K}$ preserves isofibrations

The models I gave for $(\infty,1)$-categories are all Cartesian closed (not all the other examples are).

Proposition 6.2. *If $\mathcal{K}$ is Cartesian-closed, then so is $h\mathcal{K}$.*

Proof. To be Cartesian closed means $\operatorname{Fun}(A\times B,C)\cong\operatorname{Fun}(A,C^B)$. To have products means $\operatorname{Fun}(X,\prod A_i)\cong\prod\operatorname{Fun}(X,A_i)$. To have a terminal object means $\operatorname{Fun}(X,\mathbf{1})\cong\mathbf{1}=\Delta[0]$.

So if I have this then $h\operatorname{Fun}(A\times X,C)\cong h\operatorname{Fun}(A,C^B)$ and similarly in the other case. So this homotopy 2-category sometimes reflects things. □

So now we have two notions of equivalence, equivalences of ∞-categories and equivalences in the 2-category.

Theorem 6.2. *For $A\xrightarrow{f} B$ in $\mathcal{K}$ the following are equivalent and characterize equivalences.*

- **representable:** *for all X, $\operatorname{Fun}(X,A)\xrightarrow{\sim}\operatorname{Fun}(X,B)$ in quasi-categories*
- **2-categorical:** *there exists $B\xrightarrow{g}$ and natural isomorphisms $\mathrm{id}_A\xrightarrow{\alpha} gf$ and $fg\xrightarrow{\beta}\mathrm{id}_B$ in $\mathcal{K}$*
- **homotopy equivalence:** *There exist $B\xrightarrow{g} A$ and a map*

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u] \arrow[d] \\ & A \end{tikzcd}$$

and similarly for β.

Proof. To show that representability gives 2-categorical equivalence, take

$$h\,\mathrm{Fun}(B,A) \to h\,\mathrm{Fun}(B,B)$$

taking g to fg and likewise $h\,\mathrm{Fun}(A,A) \to h\,\mathrm{Fun}(A,B)$ taking gf to fgf. By full faithfulness [unintelligible].

The second implying the third uses a result of Joyal, that every isomorphism in the homotopy category of a quasi-category is homotopy coherent, so αlifts to $\mathbb{I} \xrightarrow{\alpha} \mathrm{Fun}(A,A)$ and similarly for β.

The final equivalence is the easiest, since $\mathrm{Fun}(X,-)$ preserves simplical cotensors. □

[Is one implies two the Yoneda lemma?]

Everything is the Yoneda lemma, so yeah.

Proposition 6.3. *This is what justifies the name isofibration, if I have an isofibration $E \overset{p}{\twoheadrightarrow} B$ then y is an isofibration in $h\mathcal{K}$.*

This means that if I have β a 2-cell filling

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "b"'] & E \arrow[d, two heads, "P"] \\ & B \end{tikzcd}$$

then there is a lift of b through p and of β to a 2-cell $X \to E$ which whiskers along p to β. The proof is an exercise.

So

Definition 6.5. Any ∞-category A has an ∞-category of arrows $A^{\mathbf{2}}$ with a *generic arrow* $A^{\mathbf{2}}$ with two isofibrations (domain and codomain) to A with a natural transformation K. Specifically the identity of $A^{\mathbf{2}}$ trasposes to something in $\mathrm{Fun}(X,A)^{\mathbf{2}}$ which is K, and its class is in $h\mathcal{K}$.

In the homotopy two-category, there are some subtleties.

Proposition 6.4.
(1) *for any α a natural transformation from X to A there is a map $X \xrightarrow{\lceil\alpha\rceil} A^{\mathbf{2}}$ which whiskers with the generic arrow to yield α.*
(2) *Any two such are fibered isomorphic, there is an isomorphism γ between $\lceil\alpha\rceil$ and $\lceil\alpha\rceil'$ with domain and codomain the identity.*
(3) *Any fibered two-cell is an isomorphism. If I have a map γ connecting two representatives of the same 2-cell, so that the domain and codomain of γ are the identity, then γ is an isomorphism.*

Since I'm expecting this to be somewhat unexpected, I want to say something about why it's true.

Proof. For any quasi-category, the functor $\mathrm{ho}(Q^{\mathbf{2}}) \to (\mathrm{ho}\,Q)^{\mathbf{2}}$ is *smothering*: it's surjective on objects, full, and reflects isomorphisms, so its fibers are connected groupoids. If we apply this to $Q = \mathrm{Fun}(X,A)$ we get the proposition. □

This will come back next time, but

Definition 6.6. Given $C \xrightarrow{f} A \xleftarrow{g} B$ in $\mathcal{K}$ then the *comma ∞-category* $\operatorname{Hom}_A(f,g)$ is the pullback

$$\begin{tikzcd} \operatorname{Hom}_A(f,g) \arrow[r, dashed, "\phi"] \arrow[d, dashed] & A^{\mathbf{2}} \arrow[d, "(\mathrm{cod},\mathrm{dom})"] \\ (C \times B) \arrow[r, "g\times f"] & A \times A \end{tikzcd}$$

In the homotopy two-category I get a natural transformation

$$\begin{tikzcd} & \operatorname{Hom}_A(f,g) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\Leftarrow \phi"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

So the last thing I'll say:

Proposition 6.5. *In $h\mathcal{K}$, a natural transformation $\chi : fb \Rightarrow gc$ for $b : X \to B$ and $c : X \to C$ is the whiskering of ϕ with a map $\lceil \chi \rceil : X \to \operatorname{Hom}_A(f,g)$ with second and third properties as before.*

[missed the last part]

7. Thomas Nikolaus: Higher Categories and algebraic K-theory II

Thank you. Last time I talked about how for $\mathcal{C}$ a symmetric monoidal ∞-category you associate $K(\mathcal{C})$ a spectrum associated, $(\mathcal{C}^{\cong})^{\mathrm{gp}}$, and we saw that $K(\mathrm{Fin}) \cong \mathbb{S}$ the sphere spectrum. But this is formal. We want to do rings. It's too hard to compute the homotopy, but can we compute the homology of the underlying space? Can we compute $H_*(\Omega^\infty K\mathcal{C})$. More generally if M is an ${}_\infty$-monoid, we want to understand the homology of the group completion.

There are a bunch of obvious observations, that $H_*(M)$ and $H_*(M^{\mathrm{gp}})$ are rings and this is a map of graded commutative rings.

We can consider $\pi_0 M \subset H_0(M)$, which is sent to units. So the statement is that this is sent to units because it goes to π_0 of the group completion, so we get a map from the localization with respect to π_0 to $H_*(M^{\mathrm{gp}})$.

Theorem 7.1 (Segal–McDuff)**.** *The induced map $H_*(M)[\pi_0 M^{-1}] \to H_*(M^{\mathrm{gp}})$ is an isomorphism of graded rings.*

Proof. Let me prove this, or at least sketch a proof. The map $M \to M^{\mathrm{gp}}$ is initial among maps with target an ${}_\infty$-group. We observe that it is also initial among maps (of ${}_\infty$-monoids) which send $\pi_0 M$ to units. A priori that's a different statement. It is easy to check that it's indeed true. Now taking homology is a left adjoint functor, $H\mathbb{Z}[-]$ is a functor from Mon_∞ to ${}_\infty$-algebras over $H\mathbb{Z}$, this is take the suspension and smash with $H\mathbb{Z}$. So this is a spectrum with homotopy this homology ring. The right adjoint is Ω^∞. It then follows formally that the map from $H\mathbb{Z}[M]$ to $H\mathbb{Z}[M^{\mathrm{gp}}]$ is initial under maps of ring spectra that invert $\pi_0 M$, because I map out of that, it's the same as mapping into Ω^∞. Now we know how to localize elements in a ring spectrum. The localization has this homology that I said. $\square$

I'm inputting something that lets you localize elements in a ring spectrum. They do something unstable, and this is a stable version.

Let me remark, there was no reason to use the integers. The second remark is that we could have worked with local coefficients, the third is that a similar thing is true if you have only an $_1$ monoid and you have a [unintelligible] condition. Are there any questions about that group completion theorem?

Let me start with an easy example? I have an ∞-category I call $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{Top}}$, which is the ∞-category associated with the topologically enriched category of finite dimensional $\mathbb{C}$-vector spaces and *isomorphisms*. The homomorphisms are GL_n. Then $K^{\mathsf{T}}(\mathbb{C})$ I can define to be $K(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{Top}})$. I'm really turning on the topology. So this is (as a topological space) the disjoint union of $BU(n)$. I'd say that this is a definition of connective K-theory, ku. Maybe this is not what you'd teach in a course.

The claim is that the underlying loop space $\Omega^\infty K^{\mathsf{T}}(\mathbb{C})$ is $BU\times\mathbb{Z}$. I claim that the canonical map $BU\times\mathbb{Z}$ which I'll write $\coprod BU \to \Omega^\infty K^{\mathrm{Top}}\mathbb{C}$, I do this category $\Omega^\infty K^{\mathsf{T}}\mathbb{C}$, and there's a point corresponding to the one dimensional vector space $\mathbb{C}$ and there's a map, this is taking a sum with $\mathbb{C}$, and so you include $BU(n)$ into $BU(n+1)$ in the standard way. So ignore what I said, this was supposed to be for the lower row.

$$\begin{tikzcd} \Omega^\infty K^{\mathrm{Top}}\mathbb{C} \arrow[r, "\oplus\mathbb{C}"] & \Omega^\infty K^{\mathrm{Top}}\mathbb{C} \arrow[r, "\oplus\mathbb{C}"] & \cdots \\ \mathrm{Vect}_{\mathbb{C}}^{\mathrm{Top}} \arrow[u] \arrow[r, "\oplus\mathbb{C}"] & \mathrm{Vect}_{\mathbb{C}}^{\mathrm{Top}} \arrow[u] \arrow[r, "\oplus\mathbb{C}"] & \cdots \arrow[u] \end{tikzcd}$$

Passing to colimits we obtain a map $BU\times\mathbb{Z}\to\Omega^\infty K^{\mathsf{T}}\mathbb{C}$. The homology of the top space is the homology of the lower space localized at the horizontal map. But $\pi_1 BU = 0$ and $\Omega^\infty K^{\mathsf{T}}\mathbb{C}$ is simple, so this is an equivalence, a homology equivalence between simple spaces.

So this is an application of group completion.

Let me make a little remark. In general there are maps $K_0R\times BGL_\infty(R)\to\Omega^\infty K(R)$ which is a homology isomorphism. This is not natural (this is important) but anyway, in fact, this is an acyclic map, so an iso with respect to all local coefficient systems pulled back from the codomain. My favorite definition of acyclic is an epimorphism in the ∞-category of spaces.

You can prove this with the stronger version of the group completion or by the universal property.

I want to relate that to Quillen's original definition of K-theory. In fact, it's a plus construction. That is, it is initial among all maps whose target is *hypoAbelian* (i.e., all fundamental groups (every basepoint) have no non-trivial perfect subgroups). In other words, it's killing all perfect subgroups. There's an ∞-category of spaces, and a full subcategory of hypoAbelian spaces and the plus construction is adjoint to that.

It's a fun exercise to connect this to other definitions.

In order to show that a map $K(R)\to K(R')$ induced from a map of rings is a p-adic equivalence it thus suffices to prove that the map $K_0R\to K_0R'$ is an isomorphism and the map $BGL_\infty R\to BGL_\infty R'$ is an $\mathbb{F}_p$-homology isomorphism.

At the end of the day if you want to show that K-theory spectra are equivalent you are going to do this thing to reduce to a group homology computation.

So here's a theorem that Suslin proved with this exact method. I mean, in some sense it's a little embarrassing, now I'm using Quillen's definition to prove theorems after introducing a fancy definition. From an abstract point of view it's much cleaner for me.

Theorem 7.2. *The map $K\mathbb{C}$ to $K^{\mathrm{Top}}\mathbb{C} = ku$ induced by $\mathrm{Proj}_{\mathbb{C}}^{\cong} \to \mathrm{Vect}_{\mathbb{C}}^{\top}$ is a p-adic equivalence for all p.*

This is, I think, an amazing result. The right thing is topological K-theory and the thing on the left is algebraic and disregards any topology. What Suslin does is shows that BGL_∞ of the complex numbers without topology to the version with topology is a $\mathbb{F}_p$ homology isomorphism.

These sorts of statement seem to be true more generally and I'm not sure why.

In turn $K_i\mathbb{C}$ for $i > 0$ are divisible groups and the torsion is $\mathbb{Q}/\mathbb{Z}$ in odd degree. In other words, there's a $\mathbb{Q}/\mathbb{Z}$ and a huge $\mathbb{Q}$-vector space. So $\mathbb{C}^\times$ as a discrete group is the units and then you have a big [unintelligible].

This is the root of the computations we can do.

Let's now come to K-theory of something like SP. I'm using the following theorem of Gabber, also a group homology argument.

Theorem 7.3. *Assume that (R, I) is a Henselian pair of commutative rings. Let me not say exactly what that means, but e.g., $I \subset R$ is a nilpotent ideal or R is I complete.*

Then the map $K(R) \to K(R/I)$ is an ℓ-adic equivalence for ℓ a prime that is invertible in R.

I won't indicate how to prove this. Let me make a remark. The condition that ℓ has to be invertible, that's a weakness. There's a version where ℓ is not necessarily invertible in R then the fiber of this map $K(R) \to K(R/I)$, ℓ-completed, is equivalent to the fiber of the map $TCR \to TC(R/I)$ (again completed).

This is due to Clauson, Mathew, Morrow, and this is cool, the failure of this theorem to be true is measured by TC.

So how does Gabber help us produce maps or understand the K-theory spectrum?

Say you want to understand $K(\mathbb{F}_p)^\wedge_\ell$. Then we can take the p-adic numbers, and you get a map $K(\mathbb{F}_p)^\wedge_\ell \leftarrow K(\mathbb{Z}_p)^\wedge_\ell$ but this $\mathbb{Z}_p$ embeds into $\mathbb{C}_p$. This $\mathbb{C}_p$ is abstractly isomorphic to $\mathbb{C}$, but this isomorphism is far from being nice. This is fractal, this inclusion into $\mathbb{C}$. So we hav a map from $K(\mathbb{F}_p)^\wedge_\ell$ into $ku^\wedge_\ell$. You can do the same thing $K(\bar{\mathbb{F}}_p) \to Ku^\wedge_\ell$. Suslin proved that this is an equivalence. You end up comparing some ring with its fraction field and for this you want to use the localization theorem of Quillen. It's not easy to state in terms of group completion K-theory.

I think it's an amazing fact. It's surprising that there is a map at all. Here's a highly structured version of this.

You can then observe, and this is what Quillen does, that $K(\mathbb{F}_p)^\wedge_\ell$, you get automatically a map to the fixed point of the Frobenius on $ku^\wedge_\ell$. It's not clear what Frobenius is supposed to be but it turns out to be the same as the pth Adams operation. It's important that we complete at ℓ. It doesn't exist integrally. Basically the Adams operation isn't a map of spectra. Maybe I should have taken the connected cover of the right thing. Here's an exercise. Compute $K(\mathbb{F}_p)$ rationally. Once you've done that, and you have taken all the ℓ-completion, you know everything except the p-torsion (but there's no p -torsion).

So we make statements about torsion and have to go through Witt vectors and Gabber's theorem and stuff like that, that's hard and it would be better not to do it.

Now I gave you maps of spectra which came from maps of rings, and I intended this to mean equivalences of spectra. These are ring spectra so these are maps of ring spectra.

Let me say a few words about ring structures on K-theory. Even though Quillen and everyone were happy with what they did with K-theory, there are some issues with the ring structures.

First let me say what happens on π_0. I have an additive and a multiplicative structure $\otimes_R : \mathrm{Proj}_R \times \mathrm{Proj}_R \to \mathrm{Proj}_R$, this is a "semi-ring category." As a result you get that $\pi_0 \mathrm{Proj}_R$ is a semi-ring, which implies that $K_0 R$ is a commutative ring. A good way to say this is that the functor which is group completion, from commutative monoids to commutative groups is a symmetric monoidal functor. In fact there's a tensor product of commutative monoids. This corespresents bilinear maps. The inclusion is not monoidal. But group completion is. By explicit models it's easy to verify. I'll state the exact same result for $_\infty$-monoids. In order to prove the analog, I'm going to say, you can't do the same argument with a formal difference. If you have found ways of getting around that Elmendorf–Mandell, May, [unintelligible], Rognes, found ways of doing this but [unintelligible].

Here's an abstract theorem I proved with Gepner and [unintelligible].

Theorem 7.4. *For every presentable Cartesian closed ∞-category D, there is a unique closed symmetric monoidal structure on the ∞ category, so what I can do is take $\mathrm{Mon}_\infty(D)$, and similarly, $\mathrm{Gp}_\infty(D)$ and also $\mathrm{Sp}(D)$, spectrum objects in D, and the claim is that these have unique symmetric monoidal structures such that the free $_\infty$ monoid, free $_\infty$-group, and free spectrum, admit symmetric monoidal structures.*

Moreover, the map from D is symmetric monoidal but internal group completion in D and the map to spectrum objects in D admit essentially unique symmetric monoidal structures.

So the tensor product I talked about before makes sense in absurd generality. The case of spectra is due, I should say, to Jacob Lurie. The first is the universal [unintelligible] category on D, the second is the universal additive category on D, and the spectra are the universal stable category on D.

Last time we chatted about Barratt–Priddy–Quillen requiring some equivalence from the $_\infty$-groups to spectra, and that's true in topoi but not more generally.

Okay, so examples. The first example is tensor product on commutative monoids or groups, these admit unique symmetric monoidal structures such that the functor, the free functor is symmetric monoidal, but since sets is generated under colimits from the point, I can say such that $\mathbb{N}$ (respectively $\mathbb{Z}$) is the units.

Of course now I get a similar statement for $C\,\mathrm{Mon}_\infty$ and $C\,\mathrm{Grp}_\infty$ and spectra which have unique $\otimes$-structures with unit $\mathrm{Fin}^{\cong}$, respectively, $\Omega^\infty\mathbb{S}$, respectively $\mathbb{S}$. Moreover the group completion and the [unintelligible] are symmetric monoidal. Moreover, $\mathrm{Sym\,Mon\,Cat}_\infty$ admits a symmetric monoidal structure such that commutative algebras are semi-ring categories.

There used to be mistakes in the literature because people expected 1-categorical statements, but there are coherence issues and the actual thing is 2-categorical.

One consequence, the K theory functor from symmetric monoidal categories into spectra inherits a canonical lax symmetric monoidal structure, so it sends rings to rings. We first go from symmetric monoidal categories to $_\infty$-monoids, which is only lax symmetric monoidal. Then we group complete et cetera.

I told you that Proj_R is a semi-ring ∞ category, and we need to complement this with the statement that for any R, we have $(\mathrm{Proj}_R, \oplus, \otimes)$ is a semi-ring ∞-category, i.e., [unintelligible].

If $(\mathcal{C}, \otimes)$ is symmetric monoidal ∞-category which has coproducts and $\otimes$ distributes over it than $(\mathcal{C}, \coprod, \otimes)$ is a semi-ring ∞-category.

Maybe I should say, it's probably over-optimistic to attribute this to us. On functors from Fin_* to anything there is an obvious structure, Day convolution, and so people probably did that before. For me, it's the main contribution, that this is unique, you can't go wrong, whatever you write down is right.

Sorry for going over time.

8. Tobias Dyckerhoff: Higher Segal spaces II

[I did not attend this talk]

9. John Francis: Factorization homology II

[I did not attend this talk]

10. Andrew Blumberg: Thirteen ways of looking at an equivariant stable category

Okay, so I want to thank the organizers for inviting me to think and for putting up with the title. I want to talk about various viewpoints about how to think about equivariant stable homotopy theory and its additive and multiplicative structure and how they interact.

This is sort of expository, I thought it would fit in nicely with this workshop, but in order to be precise about what I'm saying, I think you have to take that stuff seriously, and things I'll talk about are coming from, the ability to talk about, say, the universal property of stabilization

Boardman's definition of the stable category looks very modern. He formally stabilizes finite CW complexes and then complete with filtered colimits. This was motivated by a lot of things, but he was excoriated by Adams, and we moved to a situation, "you know it when you see it," given by Lewis–May spectra and more modern versions, say, orthogonal G-spectra, say, Mandell–May.

You take some category of spectra with G action and do things to invert Σ^V and Ω^V for V a finite-dimensional G-representation.

From this, you prove various structural properties of this category. What kind of things do you prove? You prove, say, that weak equivalences are detected, e.g., by $(\ )^H$ where H varies over subgroups of G. These are also detected by geometric fixed points $\Phi^H(\ )$. You also has [unintelligible] which tells you that for G finite, G/H is self-dual. You also have the tom Dieck splitting, [unintelligible].

One other thing to say about this perspective, another sort of theorem is that the homotopy groups form Mackey functors. In particular, you relate $\pi_0 S$ to the Burnside category, some sort of category of spans. One of the things that has happened fairly recently is to turn these things around to ask, what does it mean to think about the equivariant homotopy theory defined by this.

So for instance, we say that weak-equivalences are detected by categorical fixed points. This gives rise to a model of Sp_G in terms of spectral Mackey functors. You think of these as spectral presheaves on a spectral Burnside category, where G/H goes to the H-categorical fixed point and you keep track of the transfer in the Burnside categories.

This is tricky to get right but people have, and this is due to GUillou–May and exploited spectacularly in work of Barwick, DOtto, Glasman, Nardin, and Shah.

There's another thing you can do which is to consider what happens when you look at the fact that weak equivalences are detected by $\Phi^H(\ )$. This, you know, this is characterizing the equivariant stable category in terms of a diagram [unintelligible]. So now we look at the geometric fixed points and assemble *those* into something. I learned about this from John Greenlees, and I think there were parts written down by [unintelligible]–Kriz and Nikolaus–Schulze and Glasman and Ayala–Mazel-Gee–Rozenblyum.

So for [unintelligible], part of the Tate diagram looks like this, this is a homotopy pullback square involving the homotopy fixed points, the Tate fixed points, the geometric fixed points, and the categorical fixed points.

$$\begin{tikzcd} X^G \arrow[r] \arrow[d] & \Phi^G X \arrow[d] \\ X^{hCp} \arrow[r] & X^{tCp}. \end{tikzcd}$$

So now I want to pause and point out two things about this. Two other things to say about these, as far as I know, the spectral Mackey functors, we don't really know what to do for G a compact Lie group. The Burnside category for finite groups is intrinsically algebraic. In the compact Lie case that's not true additively. tom Dieck tells us what the Burnside category is, but it's [unintelligible].

In this one, I believe the version, it's not a big deal formally to set this up for a compact Lie group, diagrams are hard. And you can do [unintelligible] but the multiplicative structure is not so easy.

So I'll try and tell a finite story and then come back and say what we can say about the compact Lie case at the end.

So one other thing that will recur is that if you think about Γ-spaces and how Segal analyzed them, you prolong to a functor on finite CW-complexes, like excisive functors. What are these in the equivariant context? You can think about excisive functors as a model for G-spectra, and you get something like that the stable category on G is something like functors F from finite G-CW-complexes to G-spaces that are excisive in the sense that

(1) they take pushouts to pullbacks, and
(2) $F(G+\wedge_H X) \xrightarrow{z} \mathrm{Map}_H(G_+, F(X))$.

This is saying that coinduction and induction are the same. A sort of nicer way of saying this is that it's saying, sort of, something like, you know, that, we want $\bigvee_T X \to \prod_T x$ to be an equivalence. This is the sort of thing you get from basic properties of the stable category.

What I now want to do is, this is context and history, I want to say now that one thing that you want to extract from this, the G-stable category is determined essentially by choices of transfers. Somewhere here I wrote that G/H is self-dual and this self-duality arises as transfer. So whatcan we do here? And what is the multiplicative version.

At the beginning of the talk I plugged in U a universe. Gaunce Lewis proved that G/H is dualizable if and only if it embeds in the universe.

So one of the other ingredients that I want to touch on is stuff that came out of the Hill–Hopkins–Ravenel Kervaire invariant work. The stuff in the appendix in particular: the construction of a multiplicative norm N_H^G from Sp_H to Sp_G. What do we know about this? We know that $N_H^G(\mathbb{S}^{-V})$ is supposed to be $\mathbb{S}^{\mathrm{Ind}_H^G V}$. This is supposed to be lax symmetric monoidal and is supposed to have a diagonal map

$$\Phi^H X \xrightarrow{z} \Phi^G N_H^G x$$

which is an equivalence when X is suitably cofibrant. When [unintelligible] this is the p-fold smash product with the action set up in the way so that this can be true.

So you have this norm which is important for studying rings, is, well, one more thing that's true, we have a symmetric monoidal product and the norm gives a functor, and there's an adjoint.

[skipped some]

The conclusion to draw from this is that the norm construction parameterizes multiplication on commutative equivariant rings.

I now want to give an overview of work I started with Mike Hill from this point of view, using operads.

Definition 10.1. An N_∞ operad is an operad $\mathcal{O}$ in G-spaces such that

(1) $O(0) \cong *$
(2) $O(n)$ has a free Σ_n action, and
(3) $O(n)$ is a universal space for some family of subgroups of $G \times \Sigma_n$, including $\{H\} \times 1$.

A universal space has contractible fixed points for things in the family and empty fixed points for things outside the family.

There are two evident examples. One is the equivariant linear isometries operad on a complete universe, in fact on any universe. The other example is (any) G-trivial E_∞ operad. In accordance with usual conventions we'll call these "genuine" and "naive."

A reason to care about this other than sort of bloody-mindedness, maybe a reason to care, anyway the reason that we care, is an example due to McClure who points out that Tate takes genuine E_∞ to naive. So doing things that you want to do changes the multiplicative structure. This is the first example of things that happened in Hill–Hopkins about equivariant localizations.

Okay, so there's now, one thing to say, a subgroup in this family, an observation, is that an element of such a family that I'll denote $\mathcal{F}_n(O)$ arises as the graph of a homomorphism $H \to \Sigma_n$. This is an H-set structure on the set $\{1, \ldots, n\}$. So these are controlled by families of H-sets, and that's the perspective we want to take.

So the thing to say is that the collection, the homotopy category of N_∞-operads is equivalent to the poset of indexing systems, where an indexing system is, you know, is a coefficient system of subcategories, a functor out of the orbit category where G/H goes to some subcategory of H-sets which satisfy, they're closed under passage to subobjects, self-induction, so that if H/K is in $\mathcal{F}(H)$ and $T \in F(K)$ then $H \times_K T$ is in $F(H)$, finite coproducts, and contains trivial sets. This equivalence is done by Rbin and separately [unintelligible] and Gutierrez–White. These conditions look arbitrary, but if you look at the consequences for how things interact if things are G-operads. If I asked you on Wednesday during the excursion to do this, you could do that.

So this is saying you have multiplicative transfer for every admissible H. So I've been talking about, what about the additive structure? An $\mathcal{O}$-algebra in Sp_G means use a complete additive universe, an $\mathcal{O}$-algebra in Sp_G you have norm maps controlled by the associated indexing system of admissible sets. You'd expect, for instance, this must be equivalent to your [unintelligible] Burnside category, so additively, you expect that gl_1 should take you from $\mathcal{O}$-algebras in G-spectra to "$\mathcal{O}$-spectra" or whatever that is and [unintelligible] [unintelligible].

So it's worth looking at what these look like in the context of spectra, and so one thing you'd like to do, you might want a construction of a category of $\mathcal{O}$-spectra or an $\mathcal{O}$-stable category over Top_G, and this is interesting in part because N_∞ operads don't necessarily correspond to universes. I don't know how to say which representations to invert. That leads to pursuit of other kinds of models.

What do we want out of the category of $\mathcal{O}$-spectra? It has [unintelligible] isomorphisms parameterized by the indexing system. One thing you can do (this is work in some degree in progress with Dotto and Hill) is take the category of grouplike $\mathcal{O}$-algebras in Top_G (this is supposed to be the connective part) and look at the usual spectra in $\mathrm{Top}_G[\mathcal{O}]$. There's ways in which this is an unattractive model for reasons that are probably evident, but it leads to an $\mathcal{O}$-indexed version of the tom Dieck splitting. So you have the G-fixed points of what amounts to a suspension spectrum, split up.

$$(\Sigma^\infty X_+)^G \cong \bigvee_{G/H \in \mathcal{O}} \Sigma^\infty_+[\mathit{unintelligible}].$$

There is a model in terms of excisive functors. This is tricky to set up but at the end of the day you get a new model with nicer properties [unintelligible].

Once you set this up there are natural questions to ask so for instance with Basterra, Hill, Lawson, and Mandell, we're working on proving analogues of the characterization using augmented rings in this equivariant setting. There's eventual hope that this will help us with [unintelligible].

The last thing I want to do before this ends is talk about compact Lie groups (infinite ones). Everything so far is G-finite. What about ∞?

We know things will be complicated. You don't have a Burnside category. So what do you do? Before you wanted an equivalence

$$F(G_+ \wedge_H x) \to \mathrm{Maps}_H(G_+, F(X))$$

and [unintelligible] is the thing that's trivial when G is finite but not here. So that makes it harder. So I do want to say one example. In work with [unintelligible], Hill, Lawson, and Mandell, we show that $THH(R) = N_e^{S^1} R$. This was also written up in Stolz, [unintelligible]–[unintelligible]–Stolz, if R is a C_n ring, then we have another kind of norm, and this is just a cyclic bar construction, where now the action, $(q) \mapsto R\hat{}g + 1$, is the cyclic permutation with a twist.

We notice a couple of things that I want to highlight and then I'll stop.

We needed R to be a ring. But we only understand homotopy type at finite subgroups of S^1. So the thing I'll state and then conclude is that this should remind you of factorization homology, and indeed I want to describe a proposal, which is work in progress with Mike Mandell, where we define $N_H^G(R)$ as an equivariant factorization homology construction, where R is supposed to have an action of the little V-disks operad. We want to try to understand what's going on with this finite subgroup condition.

We now have ten minutes to get to Lauren's talk so maybe I'll stop.

11. July 4: Emily Riehl: The model-independent theory of $(\infty,1)$-categories III

Thanks very much. I thought I'd start with orientation. When I use the word *∞-category*, I've redefined this as a technical term to mean an object in an *∞-cosmos*, which is also a technical term, it's an axiomatic framework in which to study ∞-categories. I don't think of this as circular. The community has an idea of what ∞-category should mean, and so we've extracted and axiomatized this and we use ∞-category to mean something where these phenomena occur.

We started this course off on Monday studying adjunctions and limits and colimits in an ∞-category using 2-categorical techniques. We used this 2-category with objects ∞-categories, with morphisms ∞-functors, and 2-cells these ∞-natural transformations.

Today we'll prove the representable universal properties and give a version of an adjunction which is a fibered equivalence between hom spaces. We'll also find limits or colimits, I can describe that equivalence by saying it provides a right representation for the ∞-category of cones on diagrams over which we're taking the limit. These (adjunctions, limits) are both instances of the *comma ∞-category.* We're developing the formal category theory, we're extending this formal category theory to these higher homotopy categories. This is a pretty conservative extension. Any one-category with limits is an ∞-cosmos but there are no non-trivial natural transformation. A two-category with certain limits is as well, and our work specializes to that. Also I will, using similar techniques, give an internal characterization of Cartesian fibrations.

I'm not going to review the definition of an ∞-cosmos, because all we need is the comma ∞-category. The input to constructing a comma is three ∞ categories and a pair of functors with common codomain $C \xrightarrow{g} A \xleftarrow{f} B$. It's a pullback

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[r] \arrow[d] & A^{\mathbf{2}} \arrow[d, "\mathrm{cod,dom}"] \\ C \times B \arrow[r, "g\times f"] & A \times A \end{tikzcd}$$

This is in $\mathcal{K}$. If I look at this in the homotopy 2-category $h\mathcal{K}$, then I get

$$\begin{tikzcd} & \mathrm{Hom}_A(f,g) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[dd, phantom, "\Leftarrow \phi"] & \\ C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

and so now the proposition which is a weak universal property in the 2-category, a bijection between natural transformation as on the left and and maps over B and C as on the right below.

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ C \arrow[dr, "g"'] & \Leftarrow \chi & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \iff \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "\ulcorner \chi \urcorner"] & \\ C & & B \\ & \mathrm{Hom}_A(f,g) \arrow[ul, "\mathrm{cod}"] \arrow[ur, "\mathrm{dom}"'] & \end{tikzcd}$$

up to isomorphisms over the identities, any two are equivalent up to something that projects to the identity under domain or codomain. This goes right to left via whiskering with ϕ.

This determines $\mathrm{Hom}_A(f,g)$ in $\mathcal{K}/C \times B$ up to equivalence.

This is the bit of formal category theory that most of you haven't seen before. So let me pause for a minute.

Some commas of note, if f and g are elements x and y from 1 to A then $\mathrm{Hom}_A(x,y)$ is the *internal mapping space*. If I have a functor $B \xrightarrow{f} A$ then I can make a cospan with the identity on the left or the right, so I get two choices

$$\begin{tikzcd} & \mathrm{Hom}_A(f,A) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \phi"] & \\ A & & B \arrow[ll, "f"] \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Hom}_A(A,f) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \phi"] & \\ B \arrow[rr, "f"] & & A \end{tikzcd}$$

and these are called the *left and right representations*.

Finally, $\mathrm{Hom}_A = \mathrm{Hom}_A(A,A)$ is $A^{\mathbf{2}}$ and this is the *generic arrow* I introduced last time.

Proposition 11.1. *$B \xrightarrow{f} A$ is left adjoint to $A \xrightarrow{u} B$ if and only if $\mathrm{Hom}_A(f,A)$ is equivalent to $\mathrm{Hom}_B(B,u)$ over $A \times B$.*

Proof. Let's argue in the left to right direction, this is the classical direction. I have an adjunction, and these functors will compute the transposition of arrows.

If I start with one of my, we'll argue universally. The universal arrow ϕ in the comma cone, if I compose one the counit of the adjunction, I get something that factors through the comma cone for the left representation of f. So there's an essentially unique up to isomorphism map representing the composite on the left.

$$\begin{tikzcd} & \mathrm{Hom}_B(B,u) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \phi"] & \\ A \arrow[rr] \arrow[dr, equal] & \Leftarrow \epsilon & B \arrow[dl] \\ & A & \end{tikzcd} = \begin{tikzcd} & \mathrm{Hom}_B(B,u) \arrow[d, dashed] & \\ & \mathrm{Hom}_A(f,A) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \phi"] & \\ A & & B \arrow[ll, "f"] \end{tikzcd}$$

and I can do a similar thing in the other direction, and I just need to see that these are inverse to one another. You do a formal argument where you replace first one and then the other with the equality we were given defining these, and then you get something with a unit stacked on a counit which cancels by the triangle identity

$$\begin{tikzcd}
& \mathrm{Hom}_B(B,u) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow \phi"] & \\
A \arrow[rr, "u"] \arrow[d, equal] \arrow[dr, phantom, "\Leftarrow \epsilon"] & {} & B \arrow[d, equal] \arrow[dll, "f"'] \arrow[dl, phantom, "\Leftarrow \eta"] \\
A \arrow[rr, "u"'] & {} & B
\end{tikzcd}$$

$\square$

So I want to give three successively stronger theorems here about these.

Theorem 11.1. (1)

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow \rho"] & A
\end{tikzcd}$$

is an absolute right lifting diagram if and only if

$$\begin{tikzcd}
\mathrm{Hom}_B(B,r) \arrow[rr, "\lceil \rho \circ f(-) \rceil"] \arrow[dr] & & \mathrm{Hom}_A(f,g) \arrow[dl] \\
& C \times B &
\end{tikzcd}$$

is an equivalence.

(2) *if you don't know the lift,*

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[r, "g"] & A
\end{tikzcd}$$

admits an absolute right lifting $C \xrightarrow{r} B$ *if and only if*

$$\mathrm{Hom}_B(B,r) \cong \mathrm{Hom}_A(f,g)$$

over $C \times B$

(3) *Finally, the same diagram admits an absolute right lifting if and only if there exists a right adjoint right inverse* t *to codomain projection* $\mathrm{Hom}_A(f,g) \to C$*, and then* r *is* $\mathrm{dom}\, t$*, in other words, "if and only if* $\mathrm{Hom}_A(f,g)$ *admits a terminal element over* C*".*

Before proving this, let me roecall that ϵ from $fu \to \mathrm{id}_A$ is a counit of an adjunction if and only if it's an absolute right lifting diagram. Then we have:

Corollary 11.1. (1) *ϵ is a counit if and only if*

$$\begin{tikzcd} \mathrm{Hom}_B(B,u) \arrow[rr, "\ulcorner \epsilon \circ f(-)"] \arrow[dr] & & \mathrm{Hom}_A(f,A) \arrow[dl] \\ & A \times B & \end{tikzcd}$$

is an equivalence.

(2) *$f \dashv u$ if and only if $\mathrm{Hom}_B(B,u) \simeq_{A\times B} \mathrm{Hom}_A(f,A)$*

(3) *$f : B \to A$ admits a right adjoint if and only if there exists a right adjoint right inverse to codomain projection $\mathrm{Hom}_A(f,A) \to A$, if and only if $\mathrm{Hom}_A(f,A)$ has a terminal element over A (the universal property of the counit is that for all elements a in A, ϵ_a is terminal in $\mathrm{Hom}_A(f,a)$)*

I want to do the same thing for limits. Recall that a limit cone is an absolute right lifting diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow \lambda"] & A^J \end{tikzcd}$$

and here J is either an ∞-category in a Cartesian closed ∞-cosmos or a simplicial set such as $\Lambda^2[2]$, the pullback diagram.

I'll call $\mathrm{Hom}_{A^J}(\Delta, d)$ the *∞-category of cones over d* The justification is that this is is the object whose elements are exactly those.

Corollary 11.2. (1) *The pair (ℓ, λ) is a limit cone if and only if*

$$\mathrm{Hom}_A(A,\ell) \xrightarrow{\ulcorner \lambda \circ \Delta(-) \urcorner} \mathrm{Hom}_{A^J}(\Delta, d)$$

is an equivalence over A.

(2) *If I don't know the cone, but I have a conjecture about the element, then ℓ is the limit of $1 \xrightarrow{d} A^J$ if and only if $\mathrm{Hom}_A(A,\ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta, d)$.*

(3) *When do I have a limit at all, $1 \xrightarrow{d} A^J$ has a limit if and only if $\mathrm{Hom}_{A^J}(\Delta, d)$ has a terminal element. The universal property of the limit cone is that it's the terminal object in the ∞-category of cones.*

The rest of the time, I want to talk about Cartesian fibrations which are kind of hard to talk about in a short talk. Consider

$$\begin{tikzcd} E_b \arrow[r, "e_b"] \arrow[d] & E \arrow[d, "p"] \\ 1 \arrow[r, "b"] & B \end{tikzcd}$$

So when p is a *Cartesian fibration* then arrows in B act contravariantly on the fibers. If p is a cartesian fibration, then I can lift any 2-cell whose codomain is already lifted along p. So I cal lift $a \overset{f}{\Rightarrow} b$ to $\chi_f : e_a \Rightarrow e_b$ and there will be some extra functoriality here as well.

So this is the idea, let me give the definition. There's a choice between a 2-categorical and an internal definition in the ∞-cosmos. The 2-categorical definition is a little more familiar.

This is not just an interpretation of the version in the literature, it's a little weaker than that. So for convenience let me start with an isofibration.

Definition 11.1. (1) An isofibration $E \overset{p}{\twoheadrightarrow} B$ is a Cartesian fibration if for any 2-cell

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "b"'] \arrow[r, phantom, shift right=4, "\beta \Rightarrow"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

has a lift

$$\begin{tikzcd} X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow \chi"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that χ is a *p-Cartesian natural transformation* with a universal property I won't state but that you can look up in chapter 5 if you want.

(2) The class of p-cartesian natural transformations is stable under restriction:

$$\begin{tikzcd} Y \arrow[r, "f"] & X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Uparrow \chi"] & E \end{tikzcd}$$

is p-Cartesian.

So in particular I can do this with the codomain and domain projection along with ϕ for

$$\begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[r] \arrow[dr] & E \arrow[d] \\ & B \end{tikzcd}$$

and then the lift χ becomes a map $\lceil \chi \rceil$ from $\mathrm{Hom}_B(B,p)$ to $E^{\mathbf{2}}$ [missed some] and then you can see that the composition of $\lceil \chi \rceil$ with $\lceil p\kappa \rceil$ is the identity by the universal property.

Theorem 11.2 (Internal characterization)**.** *$E \to B$ is a Cartesian fibration if and only if $E^{\mathbf{2}} \xrightarrow{\lceil pk \rceil} \mathrm{Hom}_B(B,p)$ has a right adjoint right inverse.*

Theorem 11.3. (1) *Cartesian fibrations pull back.*
(2) *Cartesian fibration is equivalence-invariant.*

missed

Proposition 11.2. *$A^{\mathbf{2}} \xrightarrow{\mathrm{dom}} A$ is a Cartesian fibration, the adjointI'm looking for is restriction along a particular inclusion $\mathbf{3} \to \mathbf{2} \times \mathbf{2}$ and that has a left adjoint left inverse, just in* Cat*.*

So being a Cartesian fibration, I want a lift of

$$\begin{tikzcd} 1 \arrow[r, "\ulcorner y \xrightarrow{g} z \urcorner"] \arrow[dr, "x"'] \arrow[r, phantom, shift right=4ex, "f \Rightarrow"] & A^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\ & A \end{tikzcd}$$

to

$$\begin{tikzcd} 1 \arrow[r, bend left, "\ulcorner g \urcorner"] \arrow[r, bend right, dotted, "?"'] \arrow[r, phantom, "\Uparrow \chi"] & A^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\ & A \end{tikzcd}$$

and so how do we find this?

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, dashed, "g \circ f"] & y \arrow[d, "g"] \\ z \arrow[r, equal] & z \end{tikzcd}$$

12. Thomas Nikolaus, Higher Categories and algebraic K-theory III

Last time we saw that when we do algebraic K-theory at a finite prime and then complete it's the same as topological K-theory. We also do it for $\bar{\mathbb{F}}_p$, localized at ℓ (away from p). These are the topos points in the étale topos. Finally we showed that $K(R)$ is an ${}_\infty$-ring spectrum. Today I want to localize at $\ell = p$. So Quillen already considered this, with λ-rings, a way more powerful tool is trace methods, but I don't want to talk about those in this lecture series.

Now R is an actual commutative ring. We have functors Λ^n from Proj_R to itself. This takes M to $(M^{\otimes n})_{\Sigma_n}$. These give rise to operations $\lambda^n : K_0 R \to K_0 R$. There's a formula for what happens to the sums, but it's not entirely obvious how this works. Let me gloss this point and assume we just have these operations. Then we have the following theorem.

Theorem 12.1.
(1) *(Grothendieck)* $(K_0(R), \lambda^n)$ *forms a "λ-ring"—I won't say what this means precisely, the proof is not enlightening, but one consequence, this admits Adams operations ψ^p, ring maps that are Frobenius lifts, if you reduce modulo p it's Frobenius. You can recharacterize in the situation [unintelligible].*
(2) *If R is of characteristic p, then* $\psi^p = (\varphi_p)_*$
(3) *([unintelligible]and attributed to Quillen) These structures extend to higher K-groups. That is, $K_n R$ are λ-rings with* 0 *multiplication. The maps lift to the space level. This gives a cohomology theory in λ-rings. If you evaluate on the sphere you get a copy of $K_n R$ and $K_0 R$ and $K_n R$ is a zero ideal. It's a non-unital ring. SO the Frobenius is zero modulo p. So the ψ^p lifts to* 0 *modulo p.*

The proof of the third item comes to writing these down on the K-theory space level. These can't be spectra maps, these aren't stable. They satisfy the λ-relations up to homotopy. So I want to mainly lift up to the space level with a lot of coherence.

This was asked by several people. The λ-versions are just polynomials. It's not entirely clear how to give a coherent version of these operations. Then K_0 will become a consequence.

Let me first give you a corollary.

Corollary 12.1. *If R is perfect of characteristic p (i.e., Frobenius is an isomorphism) then $K_*(R)^\wedge_p$ for $* > 0$ vanish. Why is that? Since the λ-operations exist, then ψ^p is homotopy trivial mod p. Then it's also an isomorphism. Then the groups are trivial modulo p and then so after p-completion. You really want derived completion but that's what the λ-operations let you do.*

So this says that $K(\bar{\mathbb{F}}_p)^\wedge_p$ is $H\mathbb{Z}_p$. So you have the ℓ-completions, and then the Eilenberg MacLane spectrum, and so you get

$$\begin{array}{ccc} K(\bar{\mathbb{F}}_p) & \longrightarrow & \prod_{\ell \neq p} Ku^\wedge_\ell \\ \big\downarrow & & \big\downarrow \\ H\mathbb{Z} & \longrightarrow & \prod_{\ell\neq p} H\mathbb{Z}^\wedge_\ell \end{array}$$

is a pullback of $_\infty$-ring spectra. So this tells you everything.

The point is, just from the existence you can already see that higher K-groups are trivial.

So now let's pause for a second and see how the λ-ring structures arose. So we'd like to say that for a certain type of functor we get an induced map on K-theory. This tells us something about group completion that I didn't know and was astonished by.

So we have a group M and a commutative monoid A, and we have a map $f : A \to M$, a map of sets. This is called polynomial of degree $\leq n-1$ if the nth cross-effect

$$\mathrm{cr}_n f(a_1, \ldots, a_n) = \sum_{U\subset\{1,\ldots,n\}} (-1)^{|U|} f(\sum_{i\in U} a_i)$$

vanishes.

Exercise 12.1. Polynomials $\mathbb{Q} \to \mathbb{Q}$ are polynomials.

What are polynomial maps $\mathbb{Z} \to \mathbb{Z}$? These are polynomials with rational coefficients that send integers to integers, which are all $x \mapsto \sum^n a_i \binom{x}{i}$.

Here's a fact that I was amazed by when I learned it

Proposition 12.1 (Passi)**.**

$$\mathrm{Hom}(A^{\mathrm{gp}}, M) \to \mathrm{Hom}_{\mathrm{poly}}(A, M)$$

is a bijection.

This means group completion is actually universal not just for additive maps but also for polynomial maps.

Proof. (1) You can extend $\mathrm{Hom}_{\mathrm{poly}}(A, M)$ to $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[A], M)$, and this is polynomial of degree n if it passes to $\mathbb{Z}[A]/I^{n+1}$, the quotient by the power of the augmentation ideal.

(2) You have to check that $\mathbb{Z}[A]/I^{n+1} \to \mathbb{Z}[A^{\mathrm{gp}}]/I^{n+1}$ is an isomorphism, which is ture because a has an inverse,

$$\frac{1-(1-a)^{n+1}}{a}. \qquad \square$$

Again this tells me that group completion is more universal than you'd think.

So let me give polynomial functors

Definition 12.1 (Eilenberg–MacLane)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent-complete additive categories. They have direct sums, so you can take direct sum K-theory. A functor $F : \mathcal{A} \to \mathcal{B}$ is polynomial of degree less than or equal to $n-1$ if, we need to make sense of the cross effect

$$D_{A_1}, D_{A_n} F = 0$$

where $D_X F(Y) = \mathrm{fib}(F(X \oplus Y) \to F(Y))$. The fiber exists because it's idempotent complete.

The definition is maybe not really helpful. For example Λ^n from Proj_R to Proj_R is polynomial of degree n.

Similarly the symmetric powers and the tensor and direct sum functors. This captures the phenomena that show up when you do multilinear things. Any questions?

What is the point now? Almost by definition, a polynomial functor of additive categories gives rise to a functor on isomorphism classes of objects.

Corollary 12.2. *K_0 extends to polynomial functors.*

Let me denote additive ∞-categories and additive functors by $\mathrm{Cat}_\infty^{\mathrm{add}}$, so K_0 was a functor from this to $C\,\mathrm{Gp}(\mathrm{Set})$. But $\mathrm{Cat}_\infty^{\mathrm{add}}$ embeds into $\mathrm{Cat}_\infty^{\mathrm{poly}}$ and this extends K_0 to a functor to Set which makes the diagram (involving the forgetful functor) commute.

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{\mathrm{add}} \arrow[r, "K_0"] \arrow[d] & C\,\mathrm{Gp}(\mathrm{Set}) \arrow[d, "U"] \\
\mathrm{Cat}_\infty^{\mathrm{poly}} \arrow[r, dashed] & \mathrm{Set}\,.
\end{tikzcd}$$

Here's the theorem we prove, together with Barwick, Glasman, and Mathew.

Theorem 12.2. *$\Omega^\infty K$ extends to a polynomial functor. In other words:*

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{\mathrm{add}} \arrow[r, "K"] \arrow[d] & \mathrm{Sp} \arrow[d, "\Omega^\infty"] \\
\mathrm{Cat}_\infty^{\mathrm{poly}} \arrow[r, dashed, "\Omega^\infty K"] & S.
\end{tikzcd}$$

Let me make some remarks. In particular we get maps $\lambda^n : \Omega^\infty KR \to \Omega^\infty KR$, and these have been constructed before. The point is that all the techniques I used there, now we're extending from the homotopy theory to say that this is highly structured and as coherent as you want it to be.

More remarks. I'm promoting to forget the K-theory functor at the top, and replace it with the space. For every additive category, you have the functor $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which gives an $_\infty$ structure on $\Omega^\infty K(A)$, and $\otimes$ (if $\mathcal{A}$ is symmetric monoidal) make it an $_\infty$ ring spectrum.

The third thing I want to say, maybe I'm not going to say this, but there are variations for stable ∞-categories and n-excisive functors.

Now so far I've given polynomial functors that are very universal. For a concrete ring R there may be many more. They might not base change to higher rings. We want to distill the essence of those that extend to all rings.

Definition 12.2. (Friedlander–Suslin) A *strict* polynomial functor $\mathrm{Proj}^n_{\mathbb{Z}} \to \mathrm{Proj}^m_{\mathbb{Z}}$ is a scheme-enriched polynomial functor

There's a canonical enrichment over schemes in the sense of enriched category theory. Using the functor of points approach we can say what that is very explicitly, so let me just do that. That is, for every ring R a functor, a polynomial functor from $\mathrm{Proj}^n_R \xrightarrow{F_R} \mathrm{Proj}^m_R$ (everything is commutative here) so that for any map of commutative rings $R \to S$, we have equivalences

$$F_R(-) \otimes S \cong F_S(- \otimes_R S)$$

which are compatible with compositions.

By polynomial functoriality, these act on all K-theory spectra. I want to say that in a structured way and to do that I'll use ∞-algebraic theories (in the sense of Lawvere).

Definition 12.3. T_{FS} is the ∞-category with objects $\mathrm{Proj}^n_{\mathbb{Z}}$ and morphisms

$$\Omega^\infty K(\mathrm{Fun}^{\mathrm{strict}}(\mathrm{Proj}^n_{\mathbb{Z}}, \mathrm{Proj}^m_{\mathbb{Z}})).$$

One needs a kind of Waldhausen K-theory to make this make sense, or something, but let me skip this.

A *spectral λ-ring* is a product-preserving functor $\tilde{A}$ from T_{FS} to S, a model for this algebraic theory.

You might object what this has to do with spectra, and I'll address the point in one second.

The essence of what I've explained so far is that $\Omega^\infty K(R)$ is a spectral λ-ring. I'll notationally identify the functor with its value at 1 (at $\mathrm{Proj}_{\mathbb{Z}}$). There are these operations $\oplus$, this is like $B_{\mathbb{Z}} \times B_{\mathbb{Z}} \to B_{\mathbb{Z}}$, there is $\otimes : B_{\mathbb{Z}} \times B_{\mathbb{Z}} \to B_{\mathbb{Z}}$, for every $M \in B_{\mathbb{Z}}$ you get $\otimes M : B_{\mathbb{Z}} \to B_{\mathbb{Z}}$ and you get $\lambda^n : B_{\mathbb{Z}} \to B_{\mathbb{Z}}$. So the first three give an $E_\infty - K\mathbb{Z}$-algebra spectrum. This gives me the self-maps $\lambda^n : \Omega^\infty KR \to \Omega^\infty KR$. These have coherences that have been pushed into the category.

This now rests on what we can say about this category.

Theorem 12.3 (BGMN)**.**

(1) *The endomorphisms of* 1*, the spectrum*

$$K(\mathrm{Fun}^{\mathrm{strict}}(B_{\mathbb{Z}}, B_{\mathbb{Z}}))$$

is a direct sum of $K\mathbb{Z}$*s. That's the first part.*

(2) *The homotopy ring is given by* $(K_*\mathbb{Z})[e_1, e_2, \ldots]$ *with* e_i *in degree* 0 *corresponding to elementary symmetric polynomials, symmetric functions on* $K_*\mathbb{Z}$*.*

(3) *The homotopy category of this algebraic theory is exactly the algebraic theory of λ-rings, classical λ-rings. Automatically, for free, I get this structure.*

The first statement is a dévissage statement.

If you unfold this statement that the [unintelligible]is a λ-ring then this is very close to their proof, a higher version of their group.

Let me talk about one thing I find surprising

Corollary 12.3. *If I take* $K(R)$ *and base change it to the integers,* $K(R) \otimes_{K\mathbb{Z}} \mathbb{Z}$ *is a simplicial commutative λ-ring.*

This is generally a very hard constraint, usually you have like divided power operations and things.

Also, if R *is characteristic* p*, then* $K(R)_{(p)}$ *is a simplicial commutative λ-ring.*

This has concrete consequences, such as divided powers on the K-theory groups.

Now this is what I wanted to say about λ-rings. For the remainder of the time I want to prepare for next time, where I'll talk about Waldhausen K-theory.

So what's the idea? The idea is to say that K_0 of a category $\mathcal{C}$ is not only group completing $\pi_0\mathcal{C}$ but also introducing relations of the form $[A] + [C] = [B]$ for an "exact" sequence $A \rightarrowtail B \to C$.

Definition 12.4 (Waldhausen, Barwick)**.** A *Waldhausen ∞-category* is a pointed ∞-category (in Emily's sense: it has a zero object) with a class of morphisms, let me call them *cofibrations* (Clark calls them *ingressive*, sorry Clark), denoted like this $\rightarrowtail$ which contain all equivalence such that

(1) $0 \rightarrowtail A$,
(2) pushouts of cofibrations exist and are cofibrations.

The sequences we care about are cofiber sequences.

What are examples?

(1) Abelian categories with monomorphisms as cofibrations
(2) Stable ∞-categories with all maps as cofibrations
(3) Fin_* with injections,
(4) Proj_R with split monos.

These are typical situations where you want to form K-theory by breaking up these sequences.

Definition 12.5. If $\mathcal{C}$ is a Waldhausen ∞-category, then $\Omega^\infty K\mathcal{C}$ is $\Omega|S_. \mathbb{C}^{\cong}|$ where $S_n\mathcal{C}$ is the full subcategory of the functor category

$$\mathrm{Fun}(\mathrm{Fun}(\Delta^1, \Delta^n), \mathcal{C})$$

So that's $S_.$, and a priori this defines a space. Why is this a spectrum? It's a spectrum because all $S_n\mathcal{C}$ are symmetric monoidal with respect to coproduct. So bing a Waldhausen ∞-category means you have a coproduct so that $S_n\mathcal{C}^{\cong}$ is an E_∞-monoid. Also $|S_n\mathcal{C}|$ is an $_\infty$-monoid because [unintelligible]is computed with sifted colimits.

Then by an Eckmann–Hilton argument is that $\Omega|S_n\mathcal{C}|$ is an $_\infty$-group (since it has another structure) so $K\mathcal{C}$ is a spectrum.

An exercise until nexti time is to compute this and show how it breaks up these exact sequences as I said.

13. July 5: Emily Riehl: The model-independent theory of $(\infty,1)$-categories IV

Thank you for sticking with me for this week. It's been a lot pretty fast and today will be even worse. LAst time the main takeaway is that we could give many categorical otions as fibered equivalences, $f \dashv u$ if and only if $\mathrm{Hom}_A(f, A) \simeq_{A\times B} \mathrm{Hom}_B(B, u)$ and an element ℓ is the limit of d if and only if $\mathrm{Hom}_A(A, \ell) \simeq_A \mathrm{Hom}_{A^J}(\Delta, d)$, the ∞-category of cones.

The idea I want to push today is that ∞-categorical notions may be encoded as equivalences of *modules* (or profunctors or distributors) because the comma construction $C \xrightarrow{\mathrm{Hom}_A(f,g)} B$ is the prototypical example of a module which has a left action of C and a right action of B.

I need to say what I mean but let me give some motivation. I want to talk about the calculus of modules, which resemble actual modules over actual rings. The categorical interest in modules is that it's the universal home for categorical statements of a next-level of complexity, but for us it's mainly to give the proof of model-independence of models of $(\infty, 1)$-categories. I should tell you what modules are.

Definition 13.1. A *module* $A \xrightarrow{E} B$ is a "two-sided discrete fibration," something known from 2-category theory.

The data is given by a span of isofibrations $A \xleftarrow{q} E \xrightarrow{p} B$ such that three things are true

(1) p is a Cartesian fibration over A (that is, B acts on the right of E, contravariantly). So E and $A \times B$ live over A and the map (q, p) from E to $A \times B$ is a Cartesian fibration in the slice cosmos $\mathcal{K}/A$. So p is a Cartesian fibration but the lifts can be chosen to project to identities along q.
(2) q is a coCartesian fibration over B so A acts on the left of A, so $E \to A \times B$ is a coCartesian fibration in $\mathcal{K}/B$.
(3) The fibers are discrete ∞-categories (in $(\infty, 1)$-categories this would say they are ∞-groupoids) but specifically, any natural transformation living over $A \times B$ is invertible, $E \to A \times B$ is discrete in the slice over $A \times B$.

Let me mention some examples of this.

The thing I'll shortly call the *unit module* is the arrow construction, which I'll write $A \xrightarrow{\mathrm{Hom}_A} A$, as a span it looks like $A \xleftarrow{\mathrm{cod}} \mathrm{Hom}_A \xrightarrow{\mathrm{dom}}$. [some quick explanation]

Modules are stable under pullback, so the pair of modules represented by a functor $A \xrightarrow{f} B$ so I can make the pair of right and left represented modules $A \xrightarrow{\mathrm{Hom}_B(B,f)} B$ and $B \xrightarrow{\mathrm{Hom}_A(f,A)}$.

There's a version of the Yoneda lemma for these.

Theorem 13.1. *For any functor f from A to B. It's easier to think of A as 1. Anyway, for a module $A \xrightarrow{E} B$, I can consider*

$$\mathrm{Fun}_{A\times B}(\mathrm{Hom}_B(B, f), E) \xrightarrow{\mathrm{ev}_{\mathrm{id}_f}} \mathrm{Fun}_{A\times B}(A, E)$$

If A is 1 these are fibered elements, and the statement, anyway is that these are equivalences of quasi-categories.

So what I want to tell you about now is the calculus of modules. The main thing to understand is an analogy. If I consider the interactions between ∞-categories, ∞-functors, modules, and module maps, this closely resembles rings, ring homomorphisms, bimodules, and module homomorphisms.

I'll state the theorem that describes the calculus of modules and state the parallel in rings to give some context. This is the weirdest result today

Theorem 13.2. *Any ∞-cosmos $\mathcal{K}$ has a* virtual equipment *of modules* $\mathrm{Mod}(h\mathcal{K})$ *(related to the homotopy 2-category).*

The next ten minutes or so will be explaining what this entails. I'll give two examples of virtual equipment, the categorical one and the one for rings and modules.

So a virtual equipment has

objects	infinity functors	A	ring
vertical morphisms	∞-morphisms	$A \xrightarrow{f} B$	ring homomorphisms
horizontal morphisms	modules	$A \xrightarrow{E} B$	modules
cells		$\begin{tikzcd} A_0 \arrow[r, "E_0"] \arrow[d, "f"] & A_1 \arrow[r] & \cdots \arrow[r, "E_n"] & A_n \arrow[d, "g"] \\ B \arrow[rrr, "F"] \arrow[urrr, phantom, "\Downarrow\alpha"] & & & C \end{tikzcd}$	multilinear (A_0, A_n)-module map $E_1 \times \cdots \times E_n \to F(g, f)$

So special cases are module homomorphisms from $A \leftarrow E \to B$ to $A \leftarrow F \to B$. Another one is arestriction of scalars, $E(b,a)$ the pullback of E with $X \times Y$ which gives something like $X \leftarrow E(b,a) \to Y$ mapping to $A \leftarrow E \to B$. If I have a single E I get something like $A \leftarrow E \to B$ mapping by f and g to $C \leftarrow F \to D$ with α, this is α a map from E to $F(g,f)$ a map of $A \times B$-bimodules.

There's a particular interesting module with a nullary source, which is ι from the identity to Hom_A, this is A as an A-A-bimodule. So this is the identity two-cell from $A \leftarrow A \to A$ to $A \leftarrow \mathrm{Hom}_A \to A$.

So we take the wide pullback and then do an essential fibered functor. So this is a virtual double category, a categorified multicategory, As long as I end up with a unary target.

What makes this into a virtual equipment is the restriction of scalars, the things that pull back a module along a pair of functors and coCartesian unit cells. I haven't explained what (co)Cartesian means here, there are bijections between cells of various types in this virtual double category.

An exercise which is also motivation, a pair of modules E and F are equivalent if and only if E and F are vertically isomorphic. As a corollary,

Corollary 13.1. *The homotopy* 2*-category embeds into* $\mathrm{Mod}(h\mathcal{K})$ *in three different ways.*

I can faithfully represent a natural transformation as three different kinds of cells. One possibility that we won't use is to have the horizontal modules be Hom_A and Hom_B. The second option is $\mathrm{Hom}_B(B,f)$ and $\mathrm{Hom}(B,g)$. The third is $\mathrm{Hom}(g,B)$ and $\mathrm{Hom}(f,B)$, which is reversed because this is contravariant.

The main question is why did we bother with this. To answer this question I'll make a definition very natural from the point of view of virtual equipment, of *right extensions* and *right liftings*. If you've seen right Kan extensions in a 2-category, that's what you should be thinking.

The input data is a binary cell from a pair of modules $A \xrightarrow{K} B \xrightarrow{R} C$ to $A \xrightarrow{F} C$ (by the identity maps vertically), so that any cell from $KE_1 \cdots E_n$ to F, this has a unique factorization through a cell from $E_1 \cdots E_n$ to R.

Dually, if R is given a priori and K determined a posteriori, then this is a right lifting in the virtual equipment if for any sequence $E_1 \cdots E_n R \to F$ factors uniquely through a cell $E_1 \cdots E_n \to K$.

One other thing I meant to mention, there is no corresponding notion of left extension or left lifting because the directions are wrong. The virtual equipment can only have unary targets, so I can't turn around the arrows in this picture. But that's not important for the following reason.

The homotopy 2-category embeds in the virtual equipment. This relates natural transformations in the homotopy 2-category to something in the virtual equipment.

Proposition 13.1. *Say ρ is a natural transformation in $h\mathcal{K}$ from rk to f, then ρ is an absolute right lifting (respectively a* pointwise right Kan extension*) if and only if when I embed this with the covariant embedding* $\mathrm{Hom}(B,k)\,\mathrm{Hom}(C,r) \to \mathrm{Hom}(C,f)$ *is a right lifting (right extension).*

Analagously, λ from f to ℓk is an absolute left lifting (pointwise left Kan extension) if the contravariant embedding from $\mathrm{Hom}(\ell,C)\,\mathrm{Hom}(k,B) \to \mathrm{Hom}(f,C)$ *is a right extension (right lifting).*

What's really cool is that these highly relevant more complicated to define universal properties in the 2-category are captured by this more natural structure in the virtual equipment.

So what I meant about a pointwise right Kan extension is,

Definition 13.2. ρ is a pointwise right Kan extension if, well, ρ is a right extension, and also for any generalized element $X \xrightarrow{b} B$, I can take my extension diagram ρ, and form the comma cone ϕ for the cospan $X \xrightarrow{b} B \xleftarrow{k} A$ then the diagram is still a right Kan extension. So if X is 1, then rb is the limit of $\mathrm{Hom}_B(b,k) \xrightarrow{\mathrm{cod}} A \xrightarrow{f} C$.

I was asked whether you can use Kan extensions to define limits in the first lecture and the answer is absolutely but this is less elementary so I deferred it.

We looked for this virtual equipment to study these. We didn't think about the relationship and one of my students David Meyers noticed this a week ago. I'm very grateful to him.

I want to spend the last twenty minutes talking about model-independence.

The basic ideas are simple, maybe I'll be like John Francis and say that if there's a takeaway, this should be the takeaway, and if you were a little lost, this should be more down-to-earth.

So now we'll talk about changing ∞-cosmoi. I'll talk about two kinds, the bread-and-butter, and the equivalences.

Definition 13.3. A functor $\mathcal{K} \xrightarrow{F} \mathcal{L}$ is *cosmological* if it is simplicially enriched, preserves isofibrations, and preserves limits.

It's a cosmological *biequivalence* if also

- It's essentially surjective on objects up to equivalence—for any C in $\mathcal{L}$ there is A in $\mathcal{K}$ such that FA is equivalent to C.
- It's a local equivalence, for any A and B in $\mathcal{K}$ the map $\mathrm{Fun}_{\mathcal{K}}(A,B) \to \mathrm{Fun}_{\mathcal{L}}(FA,FA)$ is an equivalence of quasi-categories.

The reason it's so strict is because there are lots of examples.

So you have cosmological biequivalences among complete Segal spaces, quasi-categories, Segal categories, and [unintelligible](1-complicial sets?)

If you have an equivalence $A \xrightarrow{B}$ in $\mathcal{K}$ then $\mathcal{K}/B \xrightarrow{f^*} \mathcal{K}/A$ is a biequivalence.

If $\mathcal{K} \xrightarrow{F} 1$ then $\mathcal{K}/B \to \mathcal{L}/FB$.

Finally, if $\mathcal{K}$ is biequivalent to quasi-categories, then there's a particular functor that's always a biequivalence, called $(-)_0$ which is $\mathrm{Fun}(1,-)$ which is the underlying quasi-category functor. An abstract biequivalence (a finite zig-zag) then that one is as well.

The main result about biequivalences is the following

Theorem 13.3. *A cosmological biequivalence $\mathcal{K} \xrightarrow{F} \mathcal{L}$ induces biequivalences of homotopy 2-categories $h\mathcal{K} \xrightarrow{h} \mathcal{L}$ and virtual equipments.*

I'll explain the rest of this board explaining what this means.

This induces first an explicit bijection on equivalence classes of ∞-categories. It preserves, reflects, and creates equivalence.

Relative to this, it's local bijection on isomorphism classes of parallel functors.

Relative to this, it's a local bijection on natural transformations.

Relative this, it's a local bijection on equivalence classes of modules in the equipment. I'll give a hint about the proof. Again, if I have A and B ∞-categories in $\mathcal{K}$ and A' and B' equivalent ones in $\mathcal{L}$ then the modules are the same.

Finally, relative to this, a local bijection on module maps.

The basic idea is that I can look at $\mathcal{K}/A \times B \xrightarrow{F} \mathcal{L}/FA \times FB \to \mathcal{L}/A' \times B'$. This composite biequivalence preserves, reflects, and creates equivalences of ∞-categories. This is more or less the proof of the fourth statement which is maybe the hardest one.

What does this have to do with models of ∞-category theory? Let me state and try to justify that.

The point is that the virtual equipment is the home to state equivalence of models.

Corollary 13.2. *A cosmological biequivalence preserves, reflects, and creates all "non-evil" ∞-categorical notions.*

The justification is that any such thing can be captured in something like the virtual equipment.

So as a specific example, $\mathcal{K} \xrightarrow{\sim} \mathcal{L}$ creates Cartesian fibrations. Say you want to work with a Cartesian fibration you have in quasi-categories, but you want it in complete Segal spaces. So we have $E' \overset{p'}{\twoheadrightarrow} B'$, then we can find $F\tilde{E} \sim E'$ and $F\tilde{B} \sim B'$ and I can find $F\tilde{p}$ which is equivalent, then I replace $\tilde{p}$ with $p : E \twoheadrightarrow B$ an isofibration. Why is this Cartesian? So p' admitted a right adjoint section with counit an isomorphism but now the notion of having an adjunction is equivalence-invariant, and the left adjoint of the adjunction is the image of the analagous functor in $\mathcal{K}$ and create the adjoint using the standard 2-categorical biequivalence. There exists again something with counit an isomorphism, and so that makes it a Cartesian fibration.

I want to end with one more sophisticated example. The last thing I wanted to mention is, the most important thing about the model-independence result is that analytic theorems are transfered across biequivalent models. Analytic theorems transfer across models. There are lots of things I know how to prove using analytic arguments in quasi-categories that I don't know in other models. So let me give an example.

In quasi-categories, if I have a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ admits an absolute right lifting if and only if there is an absoluter right lifting for every element $1 \xrightarrow{c} C$.

The corollary is that this is true for any $\mathcal{K}$ equivalent to quasi-categories (via $(-)_0$ as before).

One direction is easy, if I have an absolute right lifting I get one for every element. I'll assume that working in $\mathcal{K}$, for all elements I have this absolute right lifting. Applying the cosmological biequivalence, this preserves the absolute right lifting (which is encoded in the virtual equipment) and every element of C_0 of the underlying quasi-category is isomorphic to an element that comes from $\mathcal{K}$. So up to isomorphism quantizing is the same over all elements in C or in C_0.

So thene there is an absolute right lifting from C_0 and this creates one in $\mathcal{K}$ by the process I narrated before. There is something here, r lifting r_0, and then I can uniquely pick ρ lifting my 2-cell, and then this maps to an equivalence of modules, so it's an equivalence of modules.

Maybe I'll stop there.

14. Thomas Nikolaus, Higher Categories and algebraic K-theory IV

Thank you very much. The last time we proved that K-theory of a commutative ring was a spectral λ-ring, so then we proved that the strict thing has $K\mathbb{Z}$ with these e_i generators on the underlying thing. At the end I defined what a Waldhausen ∞-category is and how to get something out of that.

So there is a construction, a map of E_∞-monoids from $\mathcal{C}^{\cong}$ to $\Omega^\infty K\mathcal{C}$. So you have $S_1\mathcal{C}$ and $S_2\mathcal{C}$ and you can just put the bar construction of $\mathcal{C}$, with respect to direct sum. The bar construction of $\mathcal{C}$. The claim is that there's a map from the bar construction to $S_. \mathcal{C}$. So (x,y) maps to $(x \mapsto x \wedge y)$. So $\Omega B\mathcal{C}^{\cong}$ is just a model for the group completion. Then you go to $\Omega^\infty K\mathcal{C}$.

So, good. To make this precise you have to be more careful than I am here. Here's a theorem. First, Waldhausen proves what is called additivity. This says that the map from $\mathcal{C}\times\mathcal{C}$ to $S_2\mathcal{C}$, which is $\mathcal{C}^{\Delta^1}$, this functor of Waldhausen categories, induces an equivalence $K\mathcal{C}\times K\mathcal{C}$ to $K(S_2\mathcal{C})$. It's not super-hard to give this a Waldhausen structure, and that's additivity. Morally this says that Waldhausen K-theory splits cofiber sequences. You can write down an inverse.

There's this beautiful theorem due to Barwick in this language, saying Waldhausen K-theory is universal with respect to this functor. More precisely, the transformation $\mathcal{C}^{\cong}$ to $\Omega^\infty K\mathcal{C}$ as functors from Waldhausen ∞-categories into $_\infty$-monoids is initial among all such functors out of $\mathcal{C}^{\cong}$ whose target satisfies additivity and is product-preserving.

This tells us precisely that Waldhausen K-theory is universal. It's a universal property for the functor. [outline]. There's a different proof given by [unintelligible]–Gepner–Tabuada, where [unintelligible] is an idempotent, and additivity then isn't going to change it. Then the localization [unintelligible] just the additive functors.

So for example you can get ring structures, or translate the story about polynomial functors. A lot of the story that we did could be paralleled. One thing that I want to do now is to define for R a ring spectrum, I've always assumed some connectivity, now I don't. Let $\mathrm{Perf}_R \subset \mathrm{Mod}_R$ be the smallest stable subcategory closed under retracts and containing R. This is a perfect module. Equivalently, you could say it's a compact one or dualizable. I like this because you take direct sums and retracts, cofibers, and cones. Everything can be broken up to projective modules. If you are actually connective this tells you why you can compare this to K-theory as before. If R is an ordinary ring, this is the enhanced derived category of perfect complexes.

Theorem 14.1 (Gillet–Waldhausen)**.** *If R is a connective ring spectrum, then the maps from, what can you do to a connective ring spectrum, then $K^{\mathrm{gp}}(\mathrm{Proj}_R)$ this goes to $K^{\mathrm{Wald}}_{\mathrm{Proj}_R}$ and this goes further to $K^{\mathrm{Wald}}(\mathrm{Perf}_R)$, and the claim is that these are equivalences.*

So we get back our old K-theory but now we can do it for arbitrary ring spectra, not only for connective ones.

Let me say a few words about the idea of the proof. Maybe I'll verbatim say a few words about the first theorem. You end up contemplating the Segal space that Tobias talked about. On $S_.$ this is levelwise a fiber sequence. It stays a fiber sequence after geometric realization. Then you have to work and show that it's actually true. That's roughly how you prove this. So then the third thing is two theorems, two equivalences. The first one is something about the construction I erased, that's it's a levelwise equivalence after group completion. If I do group completion K-theory of the upper map, it's an equivalence. You can't quite write down an inverse, but the trick is to add enough factors that you can, well, this is a very rough sketch.

The last one, on π_0 you can break things up into projective modules, this shows that the map is surjective. If you want to prove the actual theorem, this runs as follows. Say R is an actual ring. Then $D^{\mathrm{perf}}(R)$ is a stable ∞-category, then you write down $N\,\mathrm{Ch}^b(\mathrm{Proj}_R)$ and then you have those that are actually acyclic. Then there's the Waldhausen fibration theorem, which says that you get a fiber sequence in K-theory. Then by induction using additivity, you prove that K-theory of the left-hand guy is an infinite sum of KR, the middle one too this is basically clear, you break it up into projectives, if you concentrated in a fixed range, you get one less in the acyclic ones, and so you get one extra summand which passes to the colimit.

This is true in the generality of weight structures, which was worked out by a student of Andrews named Fontes. You don't have a t-structure, you don't have canonical decompositions.

So that's what I wanted to say about the abstract situation. So let me draw a picture [picture].

The purpose of today is to say what tools we have in Waldhausen K-theory.

Good. So, right. Now let me say a word about, I already sneaked in the fibration theorem. I find this most cleanly expressed in ∞-categories. So

Definition 14.1. (1) A sequence of stable ∞-categories and exact functors $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is called a Verdier-sequence if it is a fiber and cofiber sequence in $\mathrm{Cat}_\infty^{\mathrm{stab}}$ if
   (a) $p \circ i = 0$,
   (b) $\mathcal{C}$ is the kernel of p,
   (c) $\mathcal{E}$ is obtained from $\mathcal{D}$ by killing $\mathcal{C}$, it's the cofiber, but more concretely, the Dwyer–Kan localization at maps f such that the fiber lies in $\mathcal{C}$.

(2) A sequence is *exact* if it is a fiber sequence and the map $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is an equivalence up to idempotent completion. Objects in $\mathcal{E}$ might be retracts.

What are examples? There are a bunch of obvious examples

(1) $\mathcal{C} \xrightarrow{i} \mathcal{C} \oplus \mathcal{C} \xrightarrow{\mathrm{pr}} \mathcal{C}$
(2) $\mathcal{C} \xrightarrow{\mathrm{id}} \mathcal{C}^{\Delta^1} \xrightarrow{\mathrm{cofib}} \mathcal{C}$
(3) (a more important case from the point of K-theory) if R is a ring spectrum and $x \in \pi_* R$ is central, then we can consider $\mathrm{Perf}(R)$ and can localize at the element, and consider $\mathrm{Perf}(R[x^{-1}])$. A priori it's not clear that every module in the codomain arises this way but up to idempotent completion it does. So that means that x acts nilpotently on homotopy in the fiber:

$$\mathrm{Perf}(R)^{x-\mathrm{nil}} \to \mathrm{Perf}(R) \to \mathrm{Perf}(R[x^{-1}])$$

Antieau–Barthel–Gepner show that $\mathrm{Perf}(R)^{x-\mathrm{nil}}$ is Perf_A where we have $A = \mathrm{End}_R(R/X)$. In some sense this corresponds to this ring spectrum. We'd like to get rid of a bottom (negative) term in this which is what [unintelligible] sometimes lets us do.

What is the significance of these Verdier sequences to K-theory?

Theorem 14.2 (Thomasson and many others, let me not try to be fair here because I can't, [unintelligible] and Barwick has a version of this, and so do [unintelligible]–Gepner–Tabuada)**.** *For an exact sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ there is an induced fiber sequence of connective spectra $K\mathcal{C} \to K\mathcal{D} \to K\mathcal{E}$. It is a fiber sequence of spectra if the sequence was Verdier.*

Maybe not every object of $K\mathcal{E}$ comes from $K\mathcal{D}$ but up to idempotent completion it's okay. The fact that this is not surjective in K_0 I can force this to be a fiber sequence and if you do this in a universal way you get negative homotopy groups, negative K-theory.

Let me state some abstract consequences.

The functor which sends R to $K(R)$ considered as an object in connective spectra satisfies Zariski descent. Zariski open sets are localizations and so this satisfies local descent. If X is a scheme, then I can define $K(X)$ as the Zariski sheafification of X. Before this could have changed the value on affines but now it doesn't.

So but one thing is, there's no étale descent. K-theory doesn't satisfy étale descent, so I can make it, define $K^{\text{ét}}$ as the étale (Postnikov) sheafification. Let's gloss over this and say we have already understood the stalks. If ℓ is invertible on X, then I should have said K-theory of strictly Henselian local rings, we know it, essentially a form of ku. So we totally understand the stalks of this sheaf. This sort of tells me that the étale K-theory can be totally understood (after ℓ-completing) as $\tau_{\geq 0}(ku^X)^\wedge_\ell$. In general X means the étale homotopy type. So this gives a spectral sequence from étale homotopy to étale K-theory. So you should really do some other thing, maybe "invert the Bott element". The message is that you can compute this in terms of actual étale cohomology. This is away from the characteristic. At the characteristic you want to use trace methods.

Okay? Good. So, good, so, what did I say? I guess I erased the fiber sequence, unfortunately, I said contemplate perfect modules over R and [unintelligible] and then the nilpotent term, and now we're near dévissage theorems.

Theorem 14.3 (Barwick, after Neeman)**.** *If $\mathcal{C}$ is a stable ∞-category with a bounded t-structure, then $K(\mathcal{C})$ is equivalent to $K(\mathcal{C}^\heartsuit)$.*

Let me start with an example for which this is overkill. We knew $K(\mathrm{Proj}_\mathbb{Z})$ is equivalent to $K(\mathrm{Perf}_\mathbb{Z})$. But this has a t-structure. The homology of a perfect complex is a finitely generated Abelian group. Then this is equivalent to K-theory of finitely generated Abelian groups. That's the statement. More importantly you can recover a theorem of Blumberg–Mandell. Let's say you want to compute $K(KU)$ and for some reason we can compute $K(ku)$. So we use this localization sequence and see we get

$$K(\mathrm{Perf}_{ku}^{\beta-\mathrm{nil}}) \to K(ku) \to K(KU)$$

So something on the left is bounded, and as a result it's ordinary Postnikov towers. So the thing on the left is $K(\mathrm{Mod}_\mathbb{Z}^{\mathrm{fg}})$ which is $K(\mathbb{Z})$.

So this fiber sequence is beautiful.

Let me also of course make the important case that Quillen already did of localizations. Then we need the theorem that Quillen proved, if $A \subset B$ is an exact inclusion of Abelian categories, say, and every object in B has a finite filtration whose subquotients lie in A, then the inclusion induces an equivalence on K-theory of spectra. $K(\mathcal{A}) \to K(\mathcal{B})$. It turns out that technically these theorems are totally different.

Let me also demonstrate this in the most classical example. Say you want to understand $K(\mathbb{Z})$ and you can understand $K(\mathbb{Z}[1/p])$. We've seen what to do to understand étale K-theory here. On the video you can just turn back. In any case, we can understand étale K-theory. I've not explained how to relate étale K-theory and K-theory. This is an equivalence above degree 1 after going to $\mathbb{Q}_p$. Then the fiber is $K(\mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{nil}})$ but that's just $K(\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{fg},p-\mathrm{nil}})$. This has a filtration whose quotients are p-torsion. Classical Quillen dévissage tells us this is just $K(\mathcal{F}_p)$ which I say we can also understand, that's classical. So to understand K of the integers we should compute K of the rationals. That can be compared to [unintelligible] where there is a chance of computing it. Then this is almost solved. Up to the [unintelligible] conjecture this is enough then to give you K theory of the integers.

Let me end with a question and an outlook.

Question 14.1. Is there a joint generalization of the theorem of the heart and Quillen's dévissage.

Clark has some real good thoughts on this. Rognes has said that if you assume [unintelligible] then you can look at some "fiber sequences" and wonder if they are. In the example I gave before, you have some negative homotopy group and want to dévissage that away. But Antieau–Barthel–Gepner show that there can't be such a fiber sequence. This is hard, the problem for me that the things I dream of as being true is false because of their work.

Okay, for the very last thing, let me tell you what I left out. If you really want to learn about K-theory, let me say one of the big tools I didn't mention at all is motivic cohomology, which basically arose out of making a motivic quotient to K-theory, used to prove Quillen–Lichtenbaum or [unintelligible]. The map between [unintelligible] and [unintelligible] is an isomorphism in high enough degrees.

The second thing is, there's recently this brilliant work on cdh descent and the [unintelligible] conjecture, proven by Kerz–Tamme–[unintelligible]. This says K-theory satisfies some cdh descent for certain [unintelligible]-squares. [missed some] This says that negative K-theory goes down to negative the Krull dimension. This was open for quite a number of years.

The third thing I left out, one thing I stressed in the first lecture is about geometric topology, the Farrel–Jones conjecture about K-theory of group rings, the fundamental group of a manifold or A-theory, like $K(\mathbb{S}[G])$. This says I can break this up into virtually cyclic subgroups. If this is true for the group G then it proves the Borel conjecture, if you have homotopy equivalent aspherical manifolds then they are homeomorphic.

Another thing I didn't mention at all is excision. K-theory in low degrees satisfies excision for Milnor squares. It's a theorem of Suslin related to the Karoubi conjecture that you still have [unintelligible] if you're in [unintelligible]. For me this was incomprehensible. Then Tamme gave a brilliant proof using Verdier sequences of ∞-categories.

There's a version related to trace methods. We saw that $K(\mathcal{C})$ is the universal additive functor, so as soon as you get another functor that's additive, then you get a functor. So THH or TC, and you have this map and it's a theorem of many people that the map $K(\mathcal{C})$ to $TC(\mathcal{C})$ is a good approximation. There's a nice recent theorem of Clausen–Mathew–Morrow that says, for a p-complete commutative ring, p-adic ring, the étale K-theory is equivalent to $TC(R)_p^\wedge$. This is based on the work of many people.

Okay, good, thank you very much for listening, and I wrote some notes in case you care, at some point I'll upload them on my homepage. Thanks.

15. Rune Haugseng: Higher categories of higher categories

Higher categories of higher categories, and to be more precise I want to talk about the following result,

Theorem 15.1. *If we have V an E_n-monoidal ∞-category that is compatible with small colimits, meaning it has small colimits preserved by the product in each variable, then there is an $(\infty, n+1)$-category $\mathrm{CAT}^{\mathcal{V}}_{(\infty,n)}$ of $\mathcal{V}$-enveloping (∞, n)-categories such that $\mathrm{CAT}^{\mathcal{V}}_{(\infty,n)}(1_n, 1_n)$ is $\mathrm{CAT}^{\mathcal{V}}_{(\infty,n-1)}$.*

If $\mathcal{V}$ is symmetric monoidal (or E_{m+n}-monoidal), then $\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)}$ is symmetric monodial (E_m-monoidal).

The morphisms are iterated bimodules. If functors were morphisms this would be false.

I thought I'd say something about how to construct this but before I put you all to sleep with that, as motivation, let my suggest how we might hope to harness this higher structure (laughter). Let me emphasize the *hope* here.

(1) One context would be in TQFTs, an n-dimensional extended TQFT is a symmetric monoidal functor of (∞, n)-categories

$$\mathrm{Bord}_{(\infty,n)} \to \mathcal{C}$$

and for the kind of TQFT revelant in physics and the examples from representation theory, such a TQFT should assign to the point a $\mathbb{C}$-linear $(\infty, n-1)$-category (enriched in modules over $\mathbb{C}$). This expectation can be motivated both from physics, with defect, but also with examples in low dimensions, like constructions of Turaev–Viro saying that for certain monoidal $\mathbb{C}$-linear category (a special kind of $\mathbb{C}$-linear 2-category) they give you 3-dimensional TQFTs, and special kind of braided monoidal categories (a special kind of 3-categories) you get 4-dimensional TQFTs.

Then the hope is that $\mathrm{CAT}^{\mathbb{C}}_{(\infty,n)}$ is the correct target for TQFTs.

(2) In the context of higher versions of K-theory, Baas–Dundas–Richter–Rognes showed that, they defined a version of algebraic K-theory, a very special kind of 2-category and they considered a category I'll call $2-\mathrm{Vect}$, which, morally, it's the 2-category of certain $\mathbb{C}$-linear categories, and then they showed that $K^{(2)}(2\,\mathrm{Vect})$ is closely related to $K(ku)$ and is of chromatic height 2 (due to the computation of Ausoni–Rognes).

Now based on this you might optimistically hope that you can do something similar in higher dimensions, that there is some sort of K-theory of suitable (∞, n)-categories, and that we can define $n\,\mathrm{Vect}$ as some kind of subobject of $\mathrm{Cat}^{\mathbb{C}}_{(\infty,n)}$ and then $K^{(n)}(n\,\mathrm{Vect})$ should be related to $K^n(\mathbb{C})$ and throw in the redshift conjecture and say this should have chromatic height n. This is foreshadowed by recent work of [unintelligible] where they construct a version of $K^{(2)}$ more general than what I have said here, with a secondary Chern [unintelligible].

(3) Higher Brauer groups: inside $\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)}$ you can consider invertible objects, which tensor with something to the unit, then you get a symmetric monoidal ∞-groupoid. In the case of a ring, we can call that the nth Brauer space

of the ring spectrum R, and the invertible objects in $\mathrm{Cat}_{(\infty,n)}^{\mathrm{Mod}_R}$, this delooping statement that I just erased, then $\Omega\,\mathrm{Br}_n(R) \cong \mathrm{Br}_{n-1}(R)$ so these assemble into a non-connective "Brauer spectrum" $\mathrm{BR}(R)$, so $\mathrm{Br}_0(R) \cong R^\times$ and $\mathrm{Br}_1(R) \cong \mathrm{Pic}(R)$ and $\mathrm{Br}_2(R) \cong \mathrm{Br}(R)$ and the hope I want to say is that we can say something in general, that $\mathrm{BR}(R)$ satisfies étale descent and we can compute homotopy groups as some étale cohomology groups, something like (most optimistically)

$$\pi_{-n}\,\mathrm{BR}(R) \cong H^n_{\text{ét}}(R,\mathbb{G}_m) \oplus H^{n-1}_{\text{ét}}(R,\mathbb{Z}).$$

The $n = 2$ case is due to Toën and Antieau–Gepner.

So after that motivation, let me get back to explaining the actual theorem. I'll start with, let me start by saying what I mean by enriched (∞,n)-categories, and I'll start with enriched ∞-categories, which goes back to Gepner–Haugseng. Let me start by considering the non-enriched case.

An *associative monoid* in an ∞-category $\mathcal{C}$ is a simplicial object $\Delta^{\mathrm{op}} \xrightarrow{M} \mathcal{C}$ such that $M_n \xrightarrow{\sim} M_1 \times \cdots \times M_1$ via ρ_i so that $[1] \to [n]$ with $0 \mapsto i-1$ and $1 \mapsto i$. Then $M_1 \times M_1 \xleftarrow{\sim} M_2 \xrightarrow{d_1} M_1$as the product and $* \xleftarrow{\sim} M_0 \xrightarrow{s_0} M_1$ as the unit

Now a *Segal Δ−object in $\mathcal{C}$* is a map $\Delta^{\mathrm{op}} \xrightarrow{X} \mathcal{C}$ so that $X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ via ρ_i and $[0] \to [n]$. Then Segal spaces model ∞-categories, X_0 is the space of objects, X_1 is the space of morphisms, the maps $X_1 \to X_0$ are source and target, and the composable pairs $X_1 \times_{X_0} X_1$ have a composition as before

$$X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1.$$

Now a *monoidal ∞-category* is an associative monoid in Cat_∞, so we can equivalently view that as a coCartesian fibration $V^{\otimes} \to \Delta^{\mathrm{op}}$ and dually a Cartesian fibration $V_{\otimes} \to \Delta$.

Then V will denote the underlying ∞-category, which is the fiber at $[1]$. So we'll use $V_\otimes$ to define the enrichmend, so objects there look like $(v_1,\ldots,v_n)$ with v_i in V and a morphism $(v_1,\ldots,v_n)$ to $(w_1,\ldots,w_m)$ over $\phi : [n] \to [m]$ you have

$$v_i \to \bigotimes_{\phi(i-1)<j<\phi(i)} w_j$$

So the connection to enriched categories, we can think of $(v_1,\ldots,v_n)$ as being like a category with objects 0 through n and morphisms $v_i : (i-1) \to i$. So we can present enriched categories by looking at maps from these things.

So we can define the notion of a *Segal presheaf* on $V_\otimes$ as a map $F : V_\otimes^{\mathrm{op}} \to S$ such that $F(v_1 \ldots, v_n)$ is the iterated fiber product $F(v_1) \times_{F()} F(v_2) \times_{F()} \cdots \times_{F()} F(v_n)$ via Cartesian morphisms over ρ_i and $[0] \to [n]$.

So if we restrict this presheaf to the fiber over $[1]$ this is V^{op}, and with the face maps from $[1]$ to $[0]$ we can view this as lying over $F()^2$, and the slice is $\mathrm{Fun}(F()^2, S)$, and so this is $F()^2 \to \mathcal{P}(V)$, and I'll say that F is *representable* if this functor lands in the subcategory of representable presheaves.

Then the claim is that this notion is a reasonable notion of an enriched ∞-category, representable Segal presheaves on $V_\otimes$. So say $X = F()$, the sapce of objects of the enriched ∞-category. Then $F(v)$ maps to X^2 and so I can take the fiber $F(v)_{(x,y)}$ and representability says there is an object $\mathcal{C}(x,y) \in V$ so that $F(v)_{x,y} \cong \mathrm{Map}_V(v, \mathcal{C}(x,y))$. Then for example we have a map from $F(v,w)$ to

$F(v \otimes w)$. The fiber over (x, y, z) goes to the fiber over (x, z). This goes

$$\mathrm{Map}_V(v, \mathcal{C}(x, y)) \times \mathrm{Map}_V(w, \mathcal{C}(y, z)) \to \mathrm{Map}_V(v \otimes w, \mathcal{C}(x, z)).$$

Then we see that we get a composition map taking $v = \mathcal{C}(x, y)$ and $w = \mathcal{C}(y, z)$. Push forward the identity maps. Then by the Yoneda lemma we see that [unintelligible] has to be given by this map.

This is equivalent to other ways to define enriched ∞-categories, so

$$\mathcal{P}_{\mathrm{Rep\,Seg}}(V_{\otimes}) \xrightarrow{\mathrm{ev}_{()}} S,$$

this turns out to be a Cartesian fibration and the fibers can be identified with, the fibers at X are equivalent to algebras $\mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(V)$, some kind of ∞-operad Δ_X^{op}, so the weird looking definition turns out to be the same as looking at algebras over a certain category of operads.

I was supposed to talk about enriched (∞, n)-categories. You could use Θ_n, I'll use Δ^n, and basically iterate the previous definitions. You consider a *Segal Δ^n-object* (a *Δ^n-monoid*) defined iteratively as $X : \Delta^{n,\mathrm{op}} \to \mathcal{C}$ such that $\Delta^{op}, \mathrm{Fun}(\Delta^{n-1,\mathrm{op}}, \mathcal{C})$ is a Segal object (monoid) valued in Δ^{n-1}-objects (Δ^{n-1}-monoids).

If you look at $\mathcal{C}$ being spaces, then Segal Δ^n-spoces model n-fold ∞-categories. So a Δ^n-space X, you see X_I breaks up as the limit of $X_{0,\dots,0}$ (the space of objects), and then $X_{1,0,\dots,0}, \dots, X_{0,\dots,1}$ which give you n kinds of 1-morphisms, and then by looking at things with two 1s you get a bunch of kinds of squares, until you get $X_{1,\dots,1}$, a space of n-cubes.

An *E_n-monoidal ∞-category* is a Δ^n-monoid in Cat_∞. This again gives a co-Cartesian fibration $V^{\otimes} \to \Delta n, \mathrm{op}$ and a Cartesian fibration $V_{\otimes} \to \Delta^n$. Then, in the same way as here, I can lift the Segal conditions for Δ^n to this $V_{\otimes}$ and define Segal presheaves on that. We again define a notion of representable Segal presheaves on $V_{\otimes}$, and then these now model V-enriched n-fold ∞-categories.

Proposition 15.1.

$$\mathcal{P}_{\mathrm{Rep\,Seg}}(V_{\otimes}) \to \mathcal{P}_{\mathrm{Seg}}(\Delta^n_{\langle 0 \rangle})$$

(where $\Delta^n_{\langle 0 \rangle}$ is the full subcategory on $(i_1, \dots, i_n)$ with some $i_t = 0$) is a Cartesian fibration with fibers $\mathrm{Alg}_{\Delta_X^{n,\mathrm{op}}}(\mathcal{V})$ which is a special case of work in progress with Chu.

Now we actually want to model (∞, n)-categories.

Definition 15.1 (Barwick)**.** An *n-fold Segal space* is a Segal Δ^n-space X such that $X_{0,-}$ is constant and $X_{1,-}$ is an $(n-1)$-fold Segal space.

You can reformulate this in terms of $\Delta^n_{\langle 0 \rangle}$, this only depends on, well this is instead $X|_{\Delta^{n,\mathrm{op}}_{\langle 0 \rangle}}$ is the right Kan extension of $\Delta^{n-1,\mathrm{op}} \times \{[0]\}$ and this is $(n-1)$-fold Segal.

Right, so these n-fold Segal spaces, now, are models for (∞, n)-categories. So now X_I breaks up as the limit, $X_{0,\dots,0}$ is the space of objects, we have $X_{1,0,\dots,0}$ the space of 1-morphisms (the others are trivial), up to $X_{1,\dots,1}$, the space of n-morphisms.

Now we can define in the same way, a *V-enriched (∞, n)-category* is a representable Segal presheaf on $V_{\otimes}$ such that $F|_{\Delta^{n,\mathrm{op}}_{\langle 0 \rangle}}$ satisfies the same condition as before.

I see I was quite optimistic in how much time I had. Let me say something about how to construct the $(\infty, 2)$-category of V-∞-categories. So $\Delta_{/[n]}$ has as objects sequences $(i_0, \ldots, i_k)$ with $0 \le i_0 \le \cdots \le i_k \le n$. Now we can define, just as we defined Segal objects and monoids for Δ, we can define $\Delta_{/n}$-monoids, and Segal $\Delta_{/n}$-objects as $X : \Delta^{\mathrm{op}}_{/[n]} \to \mathcal{C}$ such that

$$X(i_0, \ldots, i_k) \xrightarrow{\sim} X(i_0, i_1) \times_{X(i_1)} \cdots \times_{X(i_{k-1})} X(i_{k-1}, i_k)$$

and now $\Delta_{/[1]}$-monoids model bimodules.

More precisely such a $\Delta_{/[1]}$-monoid gives you a pair of associative monoids and a bimodule between them. If instead of monoids you take Segal spaces, then you get a kind of bimodule between categories. Then a $\Delta_{/[2]}$-monoid gives you 3 monoids $X(0,0)$ and $X(1,1)$ and $X(2,2)$ and then 3 bimodules $X(0,1)$, $X(1,2)$, and $X(0,2)$ along with an $X(1,1)$-bilinear map $X(0,1) \wedge X(1,2) \to X(0,2)$. This exhibits $X(0,2)$ as $\otimes$ if and only if X is a left Kan extension of $\Lambda_{/[n]}$ (the subcategory of $\Delta_{/[n]}$ where $i_t - i_{t-1} \le 1$). So this is a model for [unintelligible], and we can look at $\mathcal{P}_{\mathrm{Seg}}(\Delta_{/[n]})$, and look at those extended from $\Lambda_{/[n]}$, this is a double ∞-category, and you can do the enriched version by crossing over Δ with $V_\otimes$.

16. Chris Schommer-Pries: The relative tangle hypothesis

[I do not take notes at slide talks— the first half was a slide timeline of TQFTs and some motivation and questions.]

The answer to my questions, that allows us to solve these problems, is the relative tangle hypothesis.

What is this? It's an (∞, n)-variant of the Baez–Dolan tangle hypothesis. You use it to describe the passage from the $(d-1)$-theory to the d-dimensional theory. That's what makes it a relative statement, rather than an absolute statement.

Before I get to the statement of the relative tangle hypothesis, I want to go back to the Baez–Dolan tangle hypothesis, and try to come up with a proof for a particular case. So this is a warm-up.

This will be a one-category case. So $d = 1$, so the dimension of the manifolds in 1, and our $m = 2$, so we'll look at an E_2 situation. I want to consider this category $\mathrm{Cob}_1^{E_2}$, this is an ordinary category where the objects are 0-manifolds embedded in $\mathbb{R}^2$, the morphisms are 1-manifolds embedded in $\mathbb{R}^2 \times I$. [pictures]

We can compose these morphisms in a linear way and there is also an E_2 composition. There's another variant of this bordism category, $\mathrm{Cob}_0^{E_2}$, where the objects are the same and the morphisms are just braids. The morphisms have no arcs swinging back and forth and no circles, just braids.

So there's a nice functor from $\mathrm{Cob}_0^{E_2}$ to $\mathrm{Cob}_1^{E_2}$, and I'd like to describe the difference between the two and try to deduce the Baez–Dolan cobordism hypothesis. Let me make a variant to make the answer a little nicer. Now I will let the 0-manifolds have + and − and let the tangles be oriented. We can try to replace the category $\mathrm{Cob}_1^{E_2}$ with an equivalent category with a bit more structure.

I want to assume that the projection to the interval I is a Morse function. Generically it's a Morse function, let me assume this, which is a bit of a problem— even if I consider these as up-to-homotopy, then this space is not connected, I can never change the number of critical points. I'll also allow A_2 singularities, of the form $x_1^3 \pm x_2^2 \pm x_3^2 \pm \cdots$ Then consider a path on this space of Morse functions and A_2-singularities. Look at $x^3 + tx$. [pictures] We might call these Igusa functions and these are connected, this is a connected space. If we're building this as a 1-category. There are only one of these things, this is an equivalent category. Now we can assume our bordisms are equipped with these structures—and then filter.

Now the idea is to filter. This is Jacob's idea, really, to filter the bordism category. We start off with $\mathrm{Cob}_0^{E_2}$. Then in the next stage we get the part of the bordism category where we allow index 0 critical points. [pictures]. Then we allow one-handles, and birth-death and at the end we get $\mathrm{Cob}_1^{E_2}$.

A theorem of Joyal and Street on braided monoidal categories tells us that $\mathrm{Cob}_0^{E_2}$ is the free braided monoidal category on two objects + and −. At the next level we add on a space of Morse functions. [pictures]

Similarly, when we add the index one critical point, again there's a moduli space that's connected, it's again a circle. If you're careful with this analysis, at the end you get that $\mathrm{Cob}_1^{E_2}$ is free on the + point and the − point and you have a coevaluation and evaluation map, and if you compose them in either of two ways you get a straightening [picture].

So this is the free braided monoidal category on its object together with a dual for that object. Then you need a lemma to say that up to contractible choice everything is determined by the plus point.

This is similar to the strategy that Jacob outlined. Jacob's proof is a sort of symphony with three movements. There are three stages. The first part is to show that the space of Igusa functions (this is not contractible so)— the space of *framed* Igusa functions—he wants to show that this framed Igusa function space is contractible.

The second step is to use this to prove what you might call the *relative cobordism hypothesis*. This is a similar description, so you alter the cobordism category to put these Igusa categories on the top dimensional cobordism. So then he uses the associated filtration, and the relative cobordism hypothesis is what you need to do to move through this filtration.

Then there's a final act, where he shows that the relative cobordism hypothesis implies the usual one.

You can do this in order of increasing detail here. Since Jacob's paper appeared, there have been otherversions for the framed Igusa functions, Eliashberg–Mischev '11 and Sorin Galatius, and Igusa all have proofs.

The relative cobordism implying the usual cobordism hypothesis is totally rigorously done by Jacob. The last one is the relative cobordism hypothesis, after a lot of reformulation, it's Claim 3.4.12, he says that it's a claim but if you accept it, then you're done. In the paper he writes "claim", he doesn't call this a complete theorem, he's clear about that. Okay, we need to get to some homotopy theory. We need a version of the bordism category which is not just a category but involves spaces as well. We need an (∞, n)-category of bordisms.

Let me do this uniformly. So n will be the category number. m is for our E_m monoidal structure, which we'll encode in extra category directions. So d is the maximal dimension of our cobordism category. We have some sort of map $X \xrightarrow{\xi} \mathrm{Gr}_d(\mathbb{R}^{m+n})$, and we'll have a functor

$$(m,n)\,\mathrm{Bord}_d^{(X,\xi)} : (\Delta^{op})^{m+n} \to \mathrm{sSet}\,.$$

So if I give myself an $n+1$-tuple $[\vec{p}]$, I need to give a space, which will be a space of tuples $(\mathbf{t}_i)^{m+n}$ and (w^θ) and χ.

So $\mathbf{t}^i$ is a map $[p_i] \to (\mathbb{R}, \leq)$, cutting out hyperplanes.

W^d is a manifold in $\mathbb{R}^{m+n}$, and θ is an (X, ξ)-structure on W. We require that the projection (the last one) from W to $\mathbb{R}$ is an Igusa function, and that χ is a framing (I don't want to say what that is).

I have two conditions. One is that for $i \leq m$ we want that $W \cap \pi{-}1_{\{i\}}(\mathbf{t}^i_k)$ is empty.

Then for all $x \in W$ with $\pi_{\{i\}}(x) = \mathbf{t}^i_k$, we have that the projection $\pi_{\{i,i+1,\ldots,m+n\}}$ is submersive at x

[picture].

Let me state the relative cobordism hypothesis and then call it quits.

You use your generalized Morse function to filter this bordism category, you start with $\mathrm{Bord}_{d-1}^{(X,\xi)}$ and then allow yourself to have 0-handles $I_0 \mathrm{Bord}_d^{(X,\xi)}$ and then $I_1 \mathrm{Bord}_d^{(X,\xi)}$, and then you can add cusps $I_1^c \mathrm{Bord}_d^{(X,\xi)}$, and then eventually look at $\mathrm{Gr}_d(\mathbb{R}^{m+n}) = \frac{O(m+n)}{O(d)\times O(m+n-d)}$ and I can pull back X and $O(m+n-1)/O(d-k) \times O(m+n-1-d)$ over this.

The homotopy hypothesis says that there is a sequince of homotopy pushouts that looks as you might expect, of $(m+n)$-fold simplicial spaces in the Segal model structure. The first one looks like adding in a new $(m+n)$-cell. There's a moduli space which is already contained in $\mathrm{Bord}_{d-1}(x, \xi)$. For each subsequent stage there's another [unintelligible].

I'm already over time, but that's a formulation of the relative cobordism hypothesis. Then if you take the $n \to \infty$ limit, you get the relative cobordism hypothesis, but one important part is that this condition that you only need your category number to be at least two. The reason is that you're attaching handles, you don't need to go down to points, just down two, because handles have codimension two corners.

Maybe I'll stop there because I'm over time.

17. Kathryn Hess: A Künneth spectral sequence for configuration spaces of products

It's a pleasure to be here, I'd like to thank the organizers, the talk I'm going to give is a followup to a previous talk I gave here at the Newton institute five years ago. As the oldest speaker by a fair bit, my talk will be about harnessing model categories. It will be about a filtration and to get this you need to get your hands dirty and get a hold of the cofibrant replacement.

So first I'll talk about work I did with Bill Dwyer and Ben Knudsen, giving an operadic model for configuration spaces of products.

So to fix notation, I'll use the same notation as earlier this week by I don't remember whom, $\mathrm{Conf}_k(X)$ is sequences $\{(x_1, \ldots, x_k) : i \neq j \implies x_i \neq x_j\}$. Computing its homology, cohomology, homotopy type, is a hard problem. One nice thing about this space is this converts sums into products. If we look at configurations of k points in a disjoint union of two spaces, you have i points in one and $k-i$ in the other, and so you hav $\coprod_{i+j=k} \mathrm{Conf}_i(X) \times \mathrm{Conf}_j(Y)$ and then you do something with the symmetric groups.

I'm interested in a more complicated question, what about products? So Bill and Ben and I did was to provide an operadic formula, in an article that should appear this year, I think, an operadic formula for configuration spaces of k points in a product of, not two arbitrary spaces, but of parallelizable manifolds.

Now I'm going to need to explain to you what this looks like, and say what we do if we want to use this to compute anything. I sholud tell you that we haven't actually computed anything but we've got a tool that should help us.

Let me tell you what the relevant operads are. We'll be concerned primarily with an honest-to-God, old-fashioned, one-colored operad, the little m-cubes operad, $\mathcal{C}_m(k)$ is the standard embeddings of a disjoint union of k open m-cubes into the open m-cube. You've all seen the picture [picture]. The operadic structure comes from embedding cubes into cubes and erasing the outside lines.

One reason we're interested in studying it, if we look in arity k, if we look in arity k and shrink each cube to its center we get a configuration in the cube, we have a homotopy equivalence

$$\mathcal{C}_m(k) \xrightarrow{\simeq} \mathrm{Conf}_k(\square^m).$$

A variant we'll need is to look at $\mathcal{E}_m(k)$ which has in arity k the framed embeddings of k copies of $\mathbb{R}^m$ in $\mathbb{R}^m$. My choice of diffeomorphism between the open m-cube and $\mathbb{R}^m$ will give an operad map $\varphi_m : \mathcal{C}_m \to \mathcal{E}_m$, and we'll pull back a module structure along this operad map.

So we'll generalize the target, taking any m-dimensional parallelizable manifold. If M is m-dimensional parallelizable I'm going to study the following symmetric sequence $\mathcal{C}_M(k)$, where k varies. Now $\mathcal{C}_M(k)$ is framed embeddings of disjoint unions of $\mathbb{R}^m$ into M. Then a special case of this is $M = \mathbb{R}^n$. So for every k and $n_1, \dots, n_k$, we get $\mathcal{C}_M(k) \times \mathcal{E}_m(n_1) \times \cdots \times \mathcal{E}_m(n_k) \to \mathcal{C}_M(\sum n_i)$. These things are appropriately associative, unital, and equivariant.

Okay, so we're going to exploit this structure, what this tells us is that the symmetric sequence $\mathcal{C}_M$ is a right module over the operad $\mathcal{E}_m$. Because we have a morphism of operads from the m-cubes operad to $\mathcal{E}_m$, we can pull this back and see that this is a right module along the little m-cubes operad. I'll drop right because the only operads we'll talk about are right modules.

This is the important algebraic structure we're getting out of the manifold that we use to build the model of configuration spaces. One thing I should point out, just as in the case of the little cubes operad, we have a weak equivalence to configurations of k points in the cube, there's an equivalence from $\mathcal{C}_M(k)$ to configurations of k points in M.

Now suppose I have two parallelizable manifolds. So $\mathcal{C}_M$ is a module over $\mathcal{C}_m$ and $\mathcal{C}_N$ over $\mathcal{C}_n$. I want to compare to the structure for $M \times N$, for M an m-manifold and N an n-manifold, both paralellizable.

The trick is to use the Boardman–Vogt tensor product. But we have to kind of revisit it and understand maybe a little more closely what's going on.

Let's look at operads $\mathcal{O}$ and $\mathcal{P}$ in simplicial sets. Then there's an operad $\mathcal{O} \bigstar \mathcal{P}$ (there are other tensor products so I'll use this notation). Operads give you something to encode algebra structures. So what are the algebras encoded by the Boardman–Vogt tensor product, they are $\mathcal{O}$-algebras in $\mathcal{P}$-algebras or $\mathcal{P}$-algebras in $\mathcal{O}$-algebras.

I'll say a word about how one actually constructs this thing. The point is, if I want to calculate $\mathcal{O} \bigstar \mathcal{P}$, I take their coproduct, which is quie a nasty beast. This coproduct is way out there, and then quotient it by an equivalence relation, let me use some colored chalk. If I have [pictures].

The reason I made this explicit is to make a point. I needed to use a diagonal. If you're doing this in a category that's not Cartesian, you will have a problem expressing this, sets, simplicial sets, topological spaces, and so on.

I'm not feeling as athlectic as Emily, and Chris discovered the hook, so—

Why did I discuss the Boardman–Vogt tensor product, this was mentioned implicitly in Rune's talk. Essentially because products of cubes are cubes, we can get the following map of operads. There exists an operad map

$$\mathcal{C}_{n_1} \bigstar \cdots \bigstar \mathcal{C}_{n_k} \to \mathcal{C}_{n_1+\cdots+n_k}$$

so this additivity theorem

Theorem 17.1 (Dunn 1988; Brinkmeier 2000)**.** *This map is an aritywise* Σ*-equivariant weak equivalence.*

The corollary is the following for talking about configuration spaces.

Look at two operads in one arity. So look at $\mathcal{C}_m \bigstar \mathcal{C}_n \to \mathcal{C}_{m+n}$. Then let's look at what we get in arity k,

$$(\mathcal{C}_m \bigstar \mathcal{C}_n)(k) \xrightarrow{\simeq} \mathrm{Conf}_k(\square^{m+n})$$

and if you look at how $\mathcal{C}_m \bigstar \mathcal{C}_n$ is defined, it's built out of arity i and arity j points in th two pieces, so $\mathrm{Conf}_i(\square^m)$ and $\mathrm{Conf}_j(\square^n)$. So $ij = k$ because you multiply in the Boardman–Vogt tensor product.

So now we'd like to generalize this again, and think about configuration spaces in arbitrary manifolds along the same lines.

If we want to generalize to having a target other than $\mathbb{R}^m$ or $\mathbb{R}^n$. This goes back to lifting the Boardman–Vogt tensor product to modules, which I talked about here five years ago.

So we replace $\mathbb{R}^m$ by M.

We have the following theorem

Theorem 17.2 (Dwyer–H. 2014; Dwyer–H.–Knudsen 2018 (?))**.** *There is a functor*

$$\mathrm{Mod}_{\mathcal{O}} \times \mathrm{Mod}_{\mathcal{P}} \xrightarrow{-\bigstar-} \mathrm{Mod}_{\mathcal{O}\bigstar\mathcal{P}}$$

satisfying some condition on what the tensor product of free modules should look like.

It turns out that one can endow these categories with model structures where weak equivalences and fibrations are created in symmetric sequences or just sequences. With respect to the projective model structure on module categories, if I have a right $\mathcal{O}$-module which is cofibrant, then $\mathcal{M}\bigstar -$, as a functor $\mathrm{Mod}_{\mathcal{P}} \to \mathrm{Mod}_{\mathcal{O}\bigstar\mathcal{P}}$ is left Quillen. There is an explicit right adjoint. There is a similar result on the right with $\mathcal{P}$-modules. So it makes sense to talk about $\mathcal{M}\bigstar^{\mathbb{L}}\mathcal{N}$. This is, if you like, in the homotopy category.

I'll use this side board to give a generalized version of the Dunn–Brinkmeier additivity theorem. This is proved by induction from the Dunn–Brinkmeier theorem. Let ι be this aritywise equivalence $\mathcal{C}_m \bigstar \mathcal{C}_n \to \mathcal{C}_{m+n}$ and M and N as usual. Then a choice of trivialization of TM and TN gives rise to an isomorphism between, on the one hand, the derived tensor product $\mathcal{C}_M \bigstar^{\mathbb{L}} \mathcal{C}_N$ and on the other hand $\mathrm{Ho}(\iota)^*\mathcal{C}_{M\times N}$ in $\mathrm{Ho}(\mathrm{Mod}_{\mathcal{C}_m\bigstar\mathcal{C}_n})$.

I think it's a pretty result, and you could ask whether it's useful. It's somtimes helpful to decide if it's useful.

Corollary 17.1. *Look at arity k on both sides. On the right side it's equivalent to* $\mathrm{Conf}_k(M \times N)$*. On the other side it's the derived tensor product* $(\mathcal{C}_M \bigstar^{\mathbb{L}} \mathcal{C}_N)(k)$*. Maybe that doesn't feel like a whole lot of progress, but it's built from* $\mathrm{Conf}_i(M)$ *and* $\mathrm{Conf}_j(N)$ *with* $ij = k$*.*

One thing I want to mention, parallelizable manifolds are nice but you'd like to beable to reduce the structure group. A lot of that stuff works for reduction, but the missing hard piece is the skew version of the additivity theorem. There's a skew version of this, and we need someone to prove the skew version of the additivity theorem.

So now how do you do this computationally? My title gives it away—we want some sort of Tor or Künneth spectral sequence. So I'll give you the theorem and then see what I have to explain to make it make any sense.

Theorem 17.3 (H.–Knudsen)**.** *Let R be a commutative ring an $\mathcal{O}$ and $\mathcal{P}$ be simplicial operads, with $\mathcal{M}$ a right $\mathcal{O}$-module and $\mathcal{N}$ a right $\mathcal{P}$-module. Assume that everything is R-flat when I calculate homology. Then there exists a natural convergent spectral sequence that converges to what we want, which is the homology* $H_{p+q}(\mathcal{M} \bigstar^{\mathbb{L}} \mathcal{N})$*, and this should converge from* $H_p((H_* \mathcal{M} \bigstar^{\mathbb{L}} \mathcal{H}_* \mathcal{N})_q)$*.*

So we have to figure out how to get the filtration. Then we need a Boardman–Vogt product in a non-Cartesian category.

We really hope to do this when $\mathcal{M}$ is $\mathcal{C}_{S^1}$ and $\mathcal{N}$ is $\mathcal{C}_{S^1}$ and then maybe at least from E_2 on Ben thinks it may be related to the Totaro spectral sequence. This is a relatively simple way to express this, one may actually be able to do some computations.

Okay let's go back to operadic modules, there's a sort of folklore result—one place you can find it is Arone–Turchin

Proposition 17.1. *If $\mathcal{O}$ is a simplicial operad and $\mathcal{C}$ a simplicial category then* $\mathrm{Mod}_{\mathcal{O}}(\underline{\mathcal{C}})$ *is equivalent to the category of (simplicially enriched) functors*

$$\mathrm{Fun}(\mathcal{F}(\mathcal{O})^{\mathrm{op}}, \underline{\mathcal{C}}).$$

So $\mathcal{F}$ is finite sets, and the morphisms use $\mathcal{O}$. So the objects are those of finite sets, and

$$\mathcal{F}(\mathcal{O})(X, Y) = \coprod_{f \in \mathcal{F}(X,Y)} \prod \mathcal{O}(f^{-1}(y)).$$

One thing I'll remark, is suppose I start with a morphism of operads $\varphi : \mathcal{O} \to \mathcal{P}$. This gives rise to an adjunction between the module categories, which is forgetful and enriched left Kan extension, φ^* and $\varphi_!$. We also have the free $\mathcal{O}$-module functor which goes from sequences $\mathcal{C}^{\mathbb{N}}$ to $\mathrm{Mod}_{\mathcal{O}}(\mathcal{C})$, and a forgetful functor. This gives rise to a comonad with underlying functor $\mathbb{F}_{\mathcal{O}} \mathbb{U}_{\mathcal{O}}$. This is all very standard. Using this you can make bar constructions hich turns out to be important.

Proposition 17.2. *If $\mathcal{C}$ is a nice enough simplicial model category, such as simplicial sets, then the projective model structure exists on* $\mathrm{Mod}_{\mathcal{O}}(\mathcal{C})$ *exists and is simplicial.*

One version of this is due to Lynn Moser, which has something to do with earlier work I did with Emily, Brooke, and Magdalena, all fo whom are here. Then $\varphi_!$ and φ^* are a Quillen pair.

The last thing I want to say is, we have an $\mathcal{O}$-bar construction, and $\mathcal{B}_{\mathcal{O}}\mathcal{M} \xrightarrow{\cong} \mathcal{M}$ is an (explicit) cofibrant replacement.

Let's say something more about the Boardman–Vogt tensor product.

So if $\mathcal{O}$ and $\mathcal{P}$ are simplicial operads, there is a simplicial functor

$$\mu : \mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{O}\bigstar\mathcal{P})$$

which takes $(X, X') \mapsto X \times X'$ and I'll leave it as an exercise to decide what to do on morphisms.

If we suppose there is $C \times C \xrightarrow{-\otimes-} \mathcal{C}$ closed symmetric, then $\mathcal{F}(\mathcal{O})^{\mathrm{op}} \times \mathcal{F}(\mathcal{P})^{\mathrm{op}}$ goes by μ to $\mathcal{F}(\mathcal{O}\bigstar\mathcal{P})^{\mathrm{op}}$ and we have

$$\begin{tikzcd}
\mathcal{F}(\mathcal{O})^{\mathrm{op}} \times \mathcal{F}(\mathcal{P})^{\mathrm{op}} \arrow[r, "\mathcal{M}\times\mathcal{N}"] \arrow[d, "\mu"] & \mathcal{C} \times \mathcal{C} \arrow[r, "-\otimes-"] & \mathcal{C}^{\mathcal{M}\boxtimes\mathcal{N}} \\
\mathcal{F}(\mathcal{O}\bigstar\mathcal{P})^{\mathrm{op}} \arrow[urr, dashed, "\mu_!(\mathcal{M}\boxtimes\mathcal{N})=\mathcal{M}\bigstar\mathcal{N}"] & &
\end{tikzcd}$$

so this is just a left Kan extension.

So let me say how well-behaved this is. If $\mathcal{M}$ is cofibrant (with respect to this structure that we defined) then $(\mathrm{Mod}_{\mathcal{P}})_{\mathrm{proj}} \to (\mathrm{Mod}_{\mathcal{O}\bigstar\mathcal{P}})_{\mathrm{proj}}$ is left Quillen and similarly for $-\bigstar\mathcal{N}$. Then the corollary relevant for the theorem is that there is a specific model for it that gives rise to a spectral sequence.

The corollary is:

Corollary 17.2. *$\mathcal{M}\bigstar^{\mathbb{L}}\mathcal{N}$ makes sense and $\mathcal{B}_{\mathcal{O}}(\mathcal{M})\bigstar\mathcal{B}_{\mathcal{P}}(\mathcal{N})$ is a model for it.*

Corollary 17.3. *There is a spectral sequence converging to $H_*(\mathcal{M}\bigstar^{\mathbb{L}}\mathcal{N})$ given by the skeletal filtration of the diagonal of $\mathcal{B}^{\mathcal{O}}_*(\mathcal{M})\bigstar\mathcal{B}^{\mathcal{P}}_*(\mathcal{N})$.*

I have only five minutes left and I'm certainly not going over at the end of a very long very hot day so let me try to just summarize very quickly. We want to do an R-linear version of what we've done. There's no R-linear Boardman–Vogt product but there is an R-linearization of a simplical category. What does that mean? There's a functor from categories enriched in simplicial sets to categories enriched in graded R-modules given by applying homology to the hom simplicial sets. What we do then is to define what it means to look at modules over $\mathcal{O}_R$, it's not really a linearization of the operad, but a linearization of the category $\mathcal{F}(\mathcal{O})$ exists, and

Definition 17.1. Define $\mathrm{Mod}_{\mathcal{O}_R}$ to be $\mathrm{Fun}(\mathcal{F}(\mathcal{O})^{\mathrm{op}}_R, \mathrm{gr}\,\mathrm{Mod}_R)$.

Now we have a category into which we map. Now we have a result of Christensen–Hovey giving a nice model structure on $(\mathrm{gr}\,\mathrm{Mod}_R)^{\Delta^{\mathrm{op}}}$ and we put this together with Moser to get a projective model structure on $(\mathrm{Mod}_{\mathcal{O}_R})_{\mathrm{proj}}$ to get a cofibrant replacement again from the bar construction.

To conclude let me tell you how to define the Boardman–Vogt tensor product in this case. We're interested in left Kan extensions so we can do as follows. We take the product and linearize,

$$\begin{tikzcd}
(\mathcal{F}(\mathcal{O}) \times \mathcal{F}(\mathcal{P}))_R \arrow[r] \arrow[d, "\mu_R"] & \mathcal{F}(\mathcal{O})_R \times \mathcal{F}(\mathcal{P})_R \arrow[r, "\mathcal{M}\times\mathcal{N}"] & \mathrm{gr}\,\mathrm{Mod}_L^{\times 2} \arrow[r, "-\otimes-"] & \mathrm{gr}\,\mathrm{Mod}_R \\
\mathcal{F}(\mathcal{O}\bigstar\mathcal{P})_R \arrow[urrr, dashed, "(\mu_R)_!(-)=\mathcal{M}\bigstar\mathcal{N}\in\mathrm{Mod}_{(\mathcal{O}\bigstar\mathcal{P})_R}"] & & &
\end{tikzcd}$$

so we don't have to worry about doing a Boardman–Vogt structure for operads in this way, we have the module structure and that's what we need.

18. July 6: Claudia Scheimbauer: Dualizability in the higher Morita category

Thank you for coming on Friday morning after the last evening. I'll try to be gentle. I'll talk about dualizability. This is something you know so I did my homework and rewrote it. The first thing we will talk about is adjoints, we have categories $\mathcal{C}$ and $\mathcal{D}$. Then $L:\mathcal{D}\to\mathcal{C}$ is *left adjoint* to $R:\mathcal{C}\to\mathcal{D}$ if there are unit $u:\mathrm{id}_{\mathcal{D}}\to R\circ L$ and counit $v:L\circ R\to\mathrm{id}_{\mathcal{C}}$ such that $(v\circ\mathrm{id}_R)(\mathrm{id}_L\circ u):L\to L$ and $(\mathrm{id}_L\circ v)(u\circ\mathrm{id}_R)$ are identities. So now we work in any bicategory instead of CAT and just have morphisms instead of functors and bimorphisms instead of natural transformations.

Similarly, a vector space V^* is *dual* to V if there are evaluation $\mathrm{ev}_V:V^*\otimes V\to\mathbf{k}$ and $\mathrm{coev}_V:\mathbf{k}\to V\otimes V^*$ such that the two maps $V\to V$ and $V^*\to V^*$ built from these are identities. This also makes sense in any monoidal category.

So let's look at examples. So Alg has as objects algebras (over $\mathbf{k}$) and a morphism from A to B is an (A,B)-bimodule. To have this as a category I have to take isomorphism classes. Composition, given an (A,B)-bimodule M and a (B,C)-bimodule N, the composition is the relative tensor product over B. As an exercise, every object A has a dual, which is A^{op} and coevaluation and evaluation are just A viewed as a bimodule in two different ways.

What about pointed vector spaces, (V,v), so this is $\mathbf{k}\xrightarrow{v}V$. In the example before a vector space only had a dual if it was finite dimensional. Now in this context, dualizable objects are the 1-dimensional vector spaces. Somehow I reduced to the trivial thing, to the monoidal unit.

These two were examples of the dual definition, what about having an adjoint. The category of algebras is actually a bicategory, with morphisms (A,B)-bimodules and homomorphisms of bimodules. This was even a double category in Emily's talk. The claim is that ${}_AM_B$ has a left adjoint if and only if M is finitely presented and projective over A. It has a right adjoint if and only if it satisfies the same thing over B.

Let's see what we can do for higher categories.

The main place I want to go is to find an (∞,n)-generalization of this, at least for dualizability, for now.

We've seen several talks about (∞,n)-categories so far. In general I will give the definition a little informally. Informally, an (∞,n)-category has objects, 1-morphisms, 2-morphisms, and so on, up to n-morphisms and everything from $(n+1)$ onward is invertible up to isomorphism. We can truncate by discarding non-invertible higher morphisms. I'll only need the 2-truncation $\tau_{\leq 2}\mathcal{C}$ to get an $(\infty,2)$-category, and even from there one can do the homotopy category and get a bicategory $h_2(\mathcal{C})$. The other thing I can do, is if $\mathcal{C}$ is symmetric monoidal, what I want you to think about is that we can do a similar truncation to get a symmetric monoidal category $h_1(\mathcal{C})$. We have a way of, starting from a symmetric monoidal (∞,n)-category, produce a situation where we understand dualizability.

Now I'll say an object is dualizable in $\mathcal{C}$ if it has a dual in $h_1(\mathcal{C})$ and an adjoint if it has one in $h_2(\mathcal{C})$.

Definition 18.1. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. We will say $\mathcal{C}$ is fully n-dualizable (sometimes called "$\mathcal{C}$ has duals") if

(1) every object X has a dual and
(2) every k-morphism for $1 \le k < n$ has a left and right adjoint.

Now any object X in such a $\mathcal{C}$ is called n-dualizable.

Then for an arbitrary symmteric monoidal (∞, n)-category $\mathcal{C}$ there is $\mathcal{C}^{nd}$, the maximal subcategory of objects that are n-dualizable and with morphisms having left and right adjoints.

I should tell you how to get the condition for k. I can pick out the layer for k, which has $(k-1)$-morphisms, k-morphisms, and $(k+1)$-morphisms.

In Chris' talk yesterday this came up as a condition for the cobordism hypothesis and that's exactly the motivation for this result.

Let me recall our motivation, which is the cobordism hypothesis, which says that if you take any symmetric monoidal (∞, n)-category, and look at functors out of it, you can evaluate and get an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \underbrace{\mathcal{C}}_{(\infty,n)}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} (\mathcal{C}^{nd})^{\simeq}.$$

Informally, we'll look at $\mathrm{Alg}_n(S)$, the "higher Morita category" where S is a "nice" $(\infty, 1)$-category, like Ch or Vect or spaces or (nice) categories.

So in this category, objects are E_n-algebras, morphisms are bimodules (which are still E_{n-1}-algebras). The 2-morphisms are bimodules (E_{n-2}-algebras). The n-morphisms are bimodules, objects in S or I could say E_0-algebras in S. The $n+1$-morphisms are homomorphisms in S or in pointed S.

There are different ways of formally defining this. One way in this case is due to Haugseng using generalized ∞-operads. Another was part of my thesis (partly joint with Calaque). I also want to consider higher categories, so thetn I could keep going and get up to $n+k$ this way. This extension was joint with Johnson-Freyd.

I wanted to have the informal discussion to be able to state the theorem

Theorem 18.1 (Gwilliam–S.)**.** $\mathrm{Alg}_n(S)$ *in the sense of Calaque–S. is fully n-dualizable.*

Corollary 18.1. *Every object A determines a fully extended n-TFT*

$$\mathrm{Bord}_n^{\mathrm{fr}} \to \mathrm{Alg}_n(S).$$

Why is this good or bad? We got fully extended n-TFTs. It's not good because we'd like to see what we get explicitly. The recipe for that is factorization homology, so this is the A-factorization homology from John's talk. If A and B are $\mathbf{k}$ then bimodules are vector spaces and homomorphisms are linear maps. So we get vector spaces at the top, as Rune said we should.

If we think again of the example, we want vector spaces and linear maps, but we want $1+1$-dualizability, and so we'd want $n+1$-dualizability or even more.

Theorem 18.2 (Gwilliam–S.)**.** *In the pointed version of* $\mathrm{Alg}_n(S)$*, only algebras equivalent to* $\mathbf{1}$ *as E_n-algebras are dualizable.*

This is not good. For the lower dimensional part Alg_n is good but it's not good if you want to go higher.

What can we do? We should have a symmetric monoidal functor forgetting the point. Philosophically this should not be a big thing. We don't have this now, just for technical reasons. If you had this, then you get the corollary as well for the unpointed thing. So that's a conjecture.

So, questions so far?

If you look at the classical Morita bicategory. Every object has a dual. Having 2-dualizability, that's a condition. Not every morphism has a left and right adjoint. We get the lower part, every object has a dual, but we get finiteness conditions, going higher.

So examples, $S = \mathrm{Vect}$ or $S = \mathrm{Ch}$ is something to have in mind. $\mathrm{Alg}_1(S)$ for Vect gives us back what we erased. Every object is 1-dualizable. The 2-dualizable objects (let me be over $\mathbb{C}$) are finite dimensional semi-simple.

I could also choose Alg_2 of Vect, and everything is 2-dualizable, and for $n = 3$ I get the same finiteness conditions. I could instead put in $S = \mathrm{CAT}_{\mathbf{k}}$, and for the $n = 1$ case and take $\mathrm{Alg}_1(\mathrm{Fin\,Cat})$, which is Douglas–Schommer-Pries–Snyder, and every object is $2\frac{1}{2}$-dualizable. This is due to having finiteness conditions, and put in something like presentable categories. This was in Brochier–Jordan–Snyder, where they got $\mathrm{Alg}_1(\mathrm{Pr})$ is "almost 2" and $\mathrm{Alg}_2(\mathrm{Pr})$ is "almost 3"-dualizable.

This is just to kind of connect to what other people are doing.

Let me now make this a little more precise. So, what is $\mathrm{Alg}_n(S)$? Let me start with $n = 1$. First I need to start with what factorization algebras are. Let X be a topological space, I'll only need $\mathbb{R}$ or $\mathbb{R}^n$.

Definition 18.2. A *factorization algebra* on X valued in S is basically something similar to what John was talking about, a multiplicative version of a cosheaf. I have an assignment F from opens of X to S along with a map for disjoint unions of opens $U_1 \sqcup \cdots \sqcup U_n \subset V$ then I get a map $F(U_1) \otimes \cdots \otimes F(U_n) \to F(V)$ such that

(1) we have coherence
(2) we have multiplicativity $F(U \sqcup V) \leftarrow F(U) \otimes F(V)$ is a weak equivalence

This was used by Costello–Gwilliam to construct field theories. But we have simpler examples, specifically all the things I want to encode. So for A an algebra and $X = \mathbb{R}$ to every interval I assign A and I can include them into the big one, which is also A. The extra data that I have to tell you, I have the inclusion and have to give you a multiplication map, and this actually uses $\mathbb{R}$ with an orientation or a framing. The second example, I have A and B algebras and an (A, B)-bimodule M with a point, and p a point in $x = \mathbb{R}$. So for an interval left of p I assign A, for an interval right of p I assign B, and for an interval containing p I assign M. So you can think of stratified factorization homology in the sense of Ayala–Francis–Tanaka, and these are the ingredients to get there.

Theorem 18.3 (Lurie; Ayala–Francis)**.** *There is an equivalence between* $\mathrm{Alg}_{E_n}(S)$ *and certain (locally constant) factorization algebras in* S *on* $\mathbb{R}^n$*.*

This locally constancy means that the structure map for an inclusion of disks is an equivalence. In the stratified case this should be understood constructibly—respecting the stratification.

So let's do $n = 1$, $\mathrm{Alg}_1(S) = Y$ is a simplicial object, a Segal space in the sense of Toby's talk. What's my Y_0? This should be my objects of Alg_1. Here I should have objects algebras and morphisms bimodules. So I define Y_0 to be locally constant factorization algebras on $\mathbb{R}$. What will Y_1 be? I want to encode the bimodule picture. Heuristically I'll define them to be constructible factorization algebras on $\mathbb{R}$ together with a point as a stratified space. My source and target maps will be given by restricting to either side.

For a Segal space I have to tell you the higher ones, so for Y_2 I'll take two points p_1 and p_2. In the rough translation, you should think of the point as an (A, B)-bimodule. So this should be thought of, with two points, as an (A, B)-bimodule and a (B, C)-bimodule. The source and target are hopefully clear. What I have to tell you is how composition works.

So in the Segal space the composition is encoded by the d_1 map. I have my two bimodules M and N hre and the factorization algebra that lives there. I want to get a single bimodule. We can push forward factorization algebras. If I have f from $X \to Y$ and F on X, then I can define $(f_*F)(U)$ as $F(f^{-1}(U))$. So I use the map that collapses everything between M and N to a point. The gluing condition that I haven't told you about implies, if you think about the module situation, you really get the (derived) tensor product. This really gives you exactly what you want from the Morita category.

Okay, we wanted something about dualizability. Let me show you for this particular example that our theorem is true.

The proof of the first theorem for $n = 1$, we need to show that every object has a dual, and here for the other condition there is nothing to check. So let's see. We start with A my factorization algebra, and I can define a map from $\mathbb{R}$ to $\mathbb{R}$ that folds over my map and collapses. [picture]

Now I can push forward $\mathcal{F}$ under this map f and get a stratified $\mathbb{R}$. Let me redraw this. [pictures]. On the left I get $A \otimes A^{\mathrm{op}}$. If I touch the point I get A. On the right I get $\mathbf{k}$. So we produced A as an $(A^{\mathrm{op}} \otimes A, \mathbf{k})$-bimodule.

I'd just like to show you why the snake identities hold. This construction was the evaluation map. It's a 1-morphism. Similarly I can get a coevaluation by bending down [pictures].

Now the snake identities, I have to check that these identities hold downstairs, but I can use information from upstairs. I need $(\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev})$ to be the identity, and this will come from the projection of this map [picture].

This this here is very nice because you can pull at the ends and get the trivial guy. This is an explanation for why you get this identity.

If you want the n-dimensional version, you take a little $\mathbb{R}^n$ and do the same thing, fold and project. Since I'm out of time I'll stop here.

19. Markus Spitzweck: Hermitian K-theory for Waldhausen ∞-categories with genuine duality

[I did not attend this talk.]