HIGHER CATEGORIES AND TQFT: WINTER SCHOOL KIAS (HIGH1 RESORT)

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1. December 19: Sasha Voronov: Higher Categories and TQFTs I

What a great place to have a school. Thank you very much for inviting us. You may have noticed that the title of my minicourse is like the title of the school. There is one small difference, I will be talking about TQFTs whereas the school is talking about TQFT. I want to talk eventually in these lectures about the cobordism hypothesis.

String theory has made a tremendous impact on mathematics. It's enough to look at the number of fields medals that went to mathematicians who used ideas of string theory to update mathematical results. Donaldson in the 90s, Jones and Witten, Kontsevich, and [unintelligible]. The basic idea is that TQFT from the mathematical side is explained by cobordisms, while physics pays attention to vector spaces of states and correlation functions. Field theory relates these two ideas, cobordisms and linear categories.

I want to talk today about two dimensional topological quantum field theories, which explain some simple topology and simple algebra. Let me start with a definition.

Definition 1.1. A *2 dimensional TQFT* is a vector space V over $\mathbf{k}$ of characteristic zero, most often the complex numbers and a correspondence between surfaces Σ with labelled boundary $1,\ldots,m$ (the inputs) and $(\overline{1},\ldots,\overline{n})$, the outputs, to $|\Sigma\rangle$, a map $V^{\otimes m}\to V^{\otimes n}$.

This should satisfy some axioms.

1. If $\Sigma_1\cong\Sigma_2$, respecting the boundary component labels, then $|\Sigma_1\rangle = |\Sigma_2\rangle$.
2. $|\Sigma_1\sqcup\Sigma_2\rangle = |\Sigma_1\otimes\Sigma_2\rangle : V^{\otimes(m+m')}\to V^{\otimes(n+n')}$
3. If we glue two surfaces Σ_1 and Σ_2 as cobordisms [picture], then $|\Sigma_2\circ\Sigma_1\rangle = |\Sigma_2\rangle\circ|\Sigma_1\rangle$.
4. An extra condition usually assumed is that a cylinder $S^1\times[0,1]$, then the corresponding operator is the identity.

Then the folklore theorem, probably first in Dijkgraaf's thesis of 1989, is the following.

Theorem 1.1. *A two dimensional topological quantum field theory is a equivalent to a commutative Frobenius algebra.*

This notion, whatever it is, explains how this data, presented geometrically, can be interpreted algebraically. It's a very simple algebraic law satisfying certain properties.

Definition 1.2. A vector space V is a *(commutative) Frobenius algebra* if it's a finite dimensional (commutative) associative algebra with a unit along with a linear map $\theta : V \to \mathbf{k}$ such that $\theta(ab)$ defines a nondegenerate symmetric bilinear form.

Exercise 1.1. Show that these are Frobenius algebras:

(1) The matrix ring $M_n(\mathbf{k})$ with trace θ the trace.
(2) The cohomology $H^\bullet(M, \mathbf{k})$ of a closed oriented manifold, with trace evaluation on the fundamental cycle.
(3) For G a finite group, the center $Z\mathbf{k}[G]$ of the group algebra $\mathbf{k}[G]$ with $\theta(\sum \lambda_g g) = \lambda_e/|G|$

The first example is non-commutative and the second example is a graded commutative example. The third is the most standard.

Let me give you the idea of how this theorem is proven, it has quite a beautiful proof. First let's go from a TQFT to a Frobenius algebra. How can we do this? There is a vector space around, and we want to define the structure of an algebra on it. So we can take a pair of pants with two inputs and one output, and then the corresponding operator will go $V \otimes V \to V$ and we'll denote it $a \cdot b$.

The next elementary cobordism is one with two inputs and no outputs, which we'll call the inner product, this will correspond to $V \otimes V \to \mathbf{k}$. This map, we'll denote it $a \otimes b \mapsto (a, b)$, which will be our future nongdegenerate symmetric bilinear form. Next is what happens to a cap with one input, call this $\theta : V \to \mathbf{k}$. The cap with no input and one output is a map $\mathbf{k} \to V$, this, let's call the image of 1 the element e of V, and this will be our future unit element.

This is more than enough of the structure than I need, and I want to check that I get a Frobenius algebra. There are axioms like commutativity, and they all come from considering diffeomorphisms between surfaces [pictures].

Associativity comes from a different picture [pictures]. The first picture gives us $(a \cdot b) \otimes c$ and then we multiply again and get $(a \cdot b) \cdot c$. The second surface, you get $a \otimes (b \cdot c)$ and then by multiplication to $a \cdot (b \cdot c)$. Then the trilinear operators should be equal.

So we've got a commutative associative algebra. What else do we get? The unit axiom is simple, it's obtained from noticing that this [picture] is diffeomorphic to a cylinder, which gives the unit $e \cdot v = v$. It's also a right unit.

What about the trace and the non-degeneracy of the trace. How do we see that it's non-degenerate. You can see that if you take a pair of pants and cap it off, then you get something diffeomorphic to the bent cylinder, so $\theta(ab) = (a, b)$, so the inner product is redundant. I want to use it to see that we have a Frobenius algebra, that this inner product is non-degenerate, which may be done this way: [picture]. Gluing this Z-shaped Riemann surface, this is also called Zorro's property.

We get $V \to V \otimes V \otimes V$, which maps $a \to a \otimes \sum b_i \otimes c_i$. This sum comes from the shape we have, I should have put this other shape which gives an operator to $k \to V \otimes V$, defined by its distinguished element in the tensor product of V with itself. It's a finite sum, maybe represented by different linear combinations, but nevertheless. Let me use, well, the next cobordism, we need to apply the inner product to the first two arguments and keep the third intact, this is $\sum (a, b_i) c_i$, and this map should be the identity, this should be equal to a for all $a \in V$. So what kind of conclusion can we draw from this? This tells us that the choice of c_i generate the whole space. Any element a can be represented as a sum of c_i. This implies that every a in V is a linear combination of the (finite) list of c_i so the dimension of V is finite and at most n, and it also shows, well, we need to see the bilinear form is nondegenerate. If v is in ker(,), then we get $\sum(v, b_i)c_i = v$. But if it's in the kernel, then the left hand side is zero, so $v = 0$.

Now we're done with the first part. We got a Frobenius algebra. Who knows, maybe we got something better or stricter, but the converse tells us that this is exactly what we get.

The second part is regarded as elementary by physicists, but is less simple for mathematicians. Often there are references to that part of the proof of the theorem, which goes from a Frobenius algebra to a TQFT, there are proofs that use Morse theory for oriented surfaces and really get kind of complicated. Myself being somewhere between mathematics and physics I want to present a simple combinatorial proof of this theorem. Suppose we have V a Frobenius algebra, and for each Σ we want to make a linear operator. So we cut Σ into pairs of pants, cylinders, caps, and so on, using notions we get from the Frobenius algebra, and obtain the operator Σ as a composition of tensor products of the corresponding operators, using pairs of pants, cups, caps, and cylinders.

I didn't mention what corresponded to a pair of pants in the other direction. To regular pants I do the product, for cylinders the identity, for caps and cups you assign θ and e, and for the last pair of pants you apply the comultiplication that comes from identification of V with its dual, getting an operator $V^* \to V^* \otimes V^*$ and use θ or rather the inner product to identify on the other side. You can check that this is cocommutative and coassociative, it's all inherited from the structure of a TQFT.

The nontrivial part is what happens if you cut your surface in a different way, into different blocks, that you can take weird closed curves on the surface, why would the corresponding operators be equal to each other. Here is some kind of idea of how you can prove it properly.

So to a cutting you assign a sort of graph [pictures]. Suppose I have two such graphs. From simple moves, using associativity, coassociativity, and other axioms of a Frobenius algebra, we can move any graph like this representing a cut oriented surface to a unique normal form. The normal form is like this [pictures]. How can we do this? Suppose your graph is a real mess. You choose input number one, find input number 2, and move it across the vertices. Moving across vertices is possible because of the axioms. Then once you've gotten the inputs and outputs to the beginning and end in this way, you pick a loop and move all the things incident on it off of it and get one disjoint loop. In this way you start going through. If you compose an oriented surface from this graph, this corresponds to gluing a surface of genus g with $m + n$ boundary components. The surface you get doing this has the same number, two normal forms are the same means the operators are the same. So your awful graph will give you the same operator as your normal form graph. This is the proof of the statement. Any questions?

In the remaining couple of minutes,

Exercise 1.2. (1) Do a similar story but consider surfaces with boundary and corners. Then the question is what would correspond to such a thing algebraically? The anticipation is that this should be a non-commutative Frobenius algebra

(2) Verify that a two-dimensional TQFT is a monoidal functor from the category of cobordisms $\mathrm{Bord}_{(1,2)}$ to the category of vector spaces. The bordisms, the morphisms are diffeomorphism classes of oriented surfaces. The objects are compact closed one dimensional oriented manifolds, disjoint unions of circles, up to diffeomorphisms not mixing components, labelled components.

2. Andrei Căldăraru: Algebraic structures on Hochschild invariants of algebraic varieties and dg-categories I

[I do not take notes at slide talks]

3. Junwu Tu: Homotopy L_∞ spaces and its applications I

I will talk about homotopy theory of A_∞ or L_∞ spaces. I got this notion by considering families of Lagrangians in a symplectic manifold. I wanted to know how Fukaya–Oh–Ohta–Ono structures varied in families. The L_∞ case was defined already by Kevin Costello. I'll discuss what I mean by homotopy theory in the first lecture. I'll talk about L_∞ and A_∞ spaces in the second and third, and then I'll give some applications.

In classical topology, first you need the notion of object, so in algebraic topology the objects you need are topological spaces, and then you need to tell me what are the morphisms, which are continuous maps. Then the algebraic topologists came up with the notion of homotopy, a relation on the set of homomorphisms. We call two maps f_0 and f_1 homotopic if there is a map $f : X \times I \to Y$ such that $f|_{X\times 0} = f_0$ and $f|_{X\times 1} = f_1$. Now you can start to do homotopy theory.

So now I want to do some version of this, let me work with complexes of vector spaces over $\mathbf{k}$ (probably characteristic zero).

My objects will be cochain complexes (C^*, d), that is $C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} \cdots$ with $d^2 = 0$. We have morphisms maps f in each degree such that f commutes with d.

Now I want to define the homotopy relation, and this is the usual one. You have this interval in topology but in the algebraic case you don't have the interval. We say $f_0 \cong f_1$ if there exists $h : C^* \to D^{*-1}$ such that $f_1 - f_0 = dh + hd$. I want to use a different definition that is equivalent to this one but which can be generalized to more situations.

I think the idea is due to Sullivan, who wanted to define an interval in kind of the same way. I want to define $\Omega^*_{\Delta^1}$, this is $\mathbf{k}[t, dt]$, which is two copies of $\mathbf{k}[t]$ one indexed by dh. Then the differential is the de Rham differential. Now we say $f_0 \cong f_1$ if there exists a map $f : C^* \to D^* \otimes \Omega^*_{\Delta^1}$. There are evaluation maps $\Omega^*_{\Delta^1} \to \mathbf{k}$ which evaluate at 0 and at 1, so $\mathrm{ev}_0(\alpha + \beta dt) = \alpha(0)$ and $\mathrm{ev}_1(\alpha + \beta dt) = \alpha(1)$. When you look at f, you postcompose with the evaluation map, we want $\mathrm{ev}_0 \circ f = f_0$ and $\mathrm{ev}_1 \circ f = f_1$.

Let's quickly check this more geometric definition is equivalent to the algebraic one. If f_0 is algebraically homotopic to f_1, then I can write $f = (1-t)f_0 + tf_1 + hdt$. This is from C^* to $D^* \otimes \Omega^*_{\Delta^1}$. If I start with some element x, then

$$d(f(x)) = d\left[(1-t)f_0(x) + tf_1(x) + h(x)dt\right]$$

and then when we evaluate the differential, I get two parts:

$$(1-t)df_0(x) + tdf_1(x) - f_0(x)dt + f_1(x)dt + dh(x)dt.$$

On the other hand we have $(1-t)f_0(dx) + tf_1(dx) + h(dx)dt$ for $f(dx)$ and the terms match up (up to sign).

Let's look at the other way up to sign. If you have $C^* \to D^* \otimes \Omega^*_{\Delta^1}$, say we send x to $\alpha(t)(x) + \beta(t)(x)dt$. These are polynomials with values in morphisms from C^* to D^*. It must commute with the differential, so we have $d\alpha(t)(x) + \alpha'(x)dt + d\beta(t)(x)dt = \alpha(t)dx + \beta(t)dxdt$. This says that if you differentiate the family of maps, $\frac{d\alpha(t)}{dt} = [d, \beta(t)]$. I have the same problem with signs. The point is now to integrate from 0 to 1, and the d we have in the commutator is independent of t, and so we get $\int_0^1 \frac{d\alpha(t)}{dt} = [d, \int_0^1 \beta(t)]$ and so this evaluetes to $\alpha(1) - \alpha(0) = [d, h]$.

An important property of this homotopy relation is that it is an equivalence relation. You can sit down and prove this easily with homological algebra. I'll give a different proof, not so elementary. It's a relation on $\mathrm{Hom}(C^*, D^*)$. I should talk about simplicial enrichment of this morphism set. So this, I'll just give you an idea of the general theory, I only need the bottom parts of this. I'll give vertices, edges, triangles, and that's all I'll need. The vertices are maps of complexes. The one-simplices are $\mathrm{Hom}(C^*, D^* \otimes \Omega^*_{\Delta^1})$. This is like an edge connecting f_0 and f_1. Then this construction of Sullivan does not have to be Δ^1, and you define your triangles as $\mathrm{Hom}(C^*, D^* \otimes \Omega^*_{\Delta^2})$. An object here will be complicated because of the two-dimensional parameter, but it will have three vertices and then you'll have these homotopies, and a biggest dimension thing. In chain complexes you can write this down, some compatibility of these homotopies, you can integrate the algebraic thing and you get something like $h_{01} - h_{12} + h_{02} = [d, h_{012}]$, something like that, very explicit. Now stop here, the next parts are important but for my talk the higher simplices are not important.

An important theorem proven by Hinich and Getzler, when you use their theorem in this particular case, tells you that,

Theorem 3.1. *(Getzler–Hinich) In this particular case,* $\mathrm{Hom}(C^*, D^*)$ *is a* Kan complex.

I won't define this but let me say what it means for us in low dimensions. It implies the following corollary.

Corollary 3.1. *The homotopy relation on* $\mathrm{Hom}(C^*, D^*)$ *is an equivalence relation.*

How do I prove this using the so-called Kan condition?

(1) With no condition, you see that any map f is equivalent to itself, implied by taking the 1-simplex $C^* \to D^* \otimes \Omega^*_{\Delta^1}$, this is with the "homotopy" $x \mapsto f(x)$ with no t dependence.
(2) How do I use the Kan condition to show symmetry. By definition there's a path from f_0 to f_1. I always have the constant path from f_0 to f_0 itself. Then the Kan condition is a lifting condition, a horn-filling condition, that says if you remove one facet from a simplex, you can lift to the boundary and interior of the simplex. Then you get a backward map and even a homotopy for this map $f_1 \to f_0$. You see nicely that this equivalence relation corresponds to removing different edges from a 2-simplex. [Pictures].

You can define the homotopy category whenever you have this sort of situation.

Definition 3.1. $\pi_0(\mathrm{ch}_{\mathbf{k}}) \cong \mathrm{ch}_{\mathbf{k}} / \cong$.

Part of topology is to characterize isomorphisms in this homotopy category.

In algebraic topology we have the Whitehead theorem, which says that a map $f: X \to Y$ for X and Y connected CW complexes is a homotopy equivalence if and only if the induced map on all homotopy groups is an isomorphism. We know that in our case, by analogy:

Theorem 3.2. *A chain map $f: C^* \to D^*$ is an isomorphism in $\pi_0(\mathrm{ch}_{\mathbf{k}})$ if and only if $H^*(f): H^*(C^*) \xrightarrow{\cong} H^*(D^*)$.*

Now I want to consider homotopy theory of A_∞ algebras.

I'll probably only have time to make definitions, like what is an A_∞ algebra. This was introduced by Stasheff. It will look algebraic but it is very topological.

Definition 3.2. An A_∞-algebra is a graded vector space A endowed with multilinear maps $m_k : A^{\otimes k} \to A$, the degree of m_k is $2-k$, such that

$$\sum_{r+s+t=N} \pm m_{r+t+1}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

for $N \geq 0$.

Over a field, we'll assume that $m_0 = 0$.

Now N is the number of inputs, so m_s has s inputs, and the total inputs is N.

When m_0 is zero, you get $m_1 m_1 = 0$ and $|m_1| = 1$ so this is a complex. Then for m_2 you have $m_2(m_1 x, y) + m_2(x, m_1 y) = m_1 m_2(x,y)$, so this m_1 is a derivation of m_2.

There are higher compatibilities that I can't write. So this is some algebraic structure.

Definition 3.3. An A_∞ homomorphism f from A to B, two algebras, is given by $f_k : A^{\otimes k} \to B$, and now I assume $k \geq 1$. The degree is given by $1-k$. The A_∞ morphism relation says that

$$\sum_{r+s+t=N} f_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = \sum_{i_1+\cdots+i_\ell=N} m_\ell(f_{i_1} \otimes \cdots \otimes f_{i_\ell})$$

So just as in the case of chain complexes you have $m_1 f_1 = f_1 m_1$ but you actually have more, this is just the first compatibility.

Definition 3.4. So two A_∞ homomorphisms f_0 and f_1 are homotopy equivalent if there is a map $f: A \to B \otimes \Omega^*_{\Delta^1}$ which restricts appropriately.

You define $M_k(b_1 \otimes \alpha_1, \ldots, b_k \otimes \alpha_k) = m_k(b_1, \ldots, b_k)(\alpha_1 \cdots \alpha_k)$ for $k \geq 1$, and $k=1$ it's M_1 the tensor differential.

Again you can prove by the same argument that this is an equivalence relation on morphisms, and then define a homotopy category. Then $\pi_0(A_\infty)$ is $A_\infty/\cong$.

There is also a Whitehead theorem in this case, which says that if $f: A \to B$, there are a lot of names here, Kadeishvili, Kontsevich, Fukaya–Oh–Ohta–Ono in this version, this is an isomorphism in the quotient category if and only if $H^*(f_1)$ is an isomorphism.

4. Grégory Ginot: Higher Hochschild homology and factorization algebras I

Somehow my talk is going to be in between Andrei's talk and Sasha's talk, and be related to some things that Junwu talked about or will talk about. The main point is to talk about homology of higher algebras related to (topological) quantum field theory described by Sasha, higher dimensional versions of what Sasha will describe. The philosophy or the motivation behind that is that, what is going to be, factorization algebras, many people here come from symplectic topology or algebraic geometry, it's common to describe a manifold by its functions, its sheaf of functions, and in the same way I want to think of a quantum field theory as being described as some algebraic data which is going to be factorization algebras. This will encode quantum field theory, I put the topological in parentheses because then the factorization algebras will be locally constant.

A topological field theory gives invariants of closed manifolds of dimension the dimension of the field theory. This is viewing the value on a closed manifold as the value of a morphism from empty to empty.

If you want to see the classical picture, in a classical picture, or perhaps "classical" picture, the invariant will be, if you have a space X, then the cohomology of the space X is the sheaf cohomology of the constant sheaf on X.

In the same way, we will get sheaf cohomology for these factorization alegbras, the invariants produced by factorization algebra and higher Hochschild homology will be global sections of locally constant factorization algebras. This will be the idea that I'll describe but I'll start with something giving an invariant for any n corresponding to differential graded commutative algebras, this is a special kind of locally constant factorization algebra and you lose some information but it's easier to compute. This is related to what Andrei has been talking about.

Let me recall the definition of Hochschild homology that Andrei was giving this morning. I want to recall the definition of the *standard Hochschild chain complex* of a differential graded associative algebra A. So

$$C_n(A) \coloneqq A^{\otimes n+1}$$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n (a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

You can see this kind of weird thing by writing them on a circle. The multiplication is by taking consecutive pairs in all possible ways. If you put this in the circle, this is easy.

The idea of higher Hochschild homology is to do the same but to put the points on any space. To do so, you will need commutative algebras. With the circle it's okay to have an associative algebra because you know the order to multiply in. If you work more generally, you need to pay attention to the order.

Let me start with a very stupid thing, a finite set with a finite number of points. If A is a unital commutative algebra, and everything can be extended naturally to differential graded commutative algebras by extending to the standard symmetric monoidal product on chain complexes, then you can define a functor from Fin the category of finite sets to $\mathbf{k}$-vector spaces which to a set I associates $A^{\otimes I}$. To a map $f : I \to J$, you define, $A^{\otimes I} \to A^{\otimes J}$, you do $\bigotimes a_i \to f_*(\bigotimes a_i) = \bigotimes b_j$ where you look at the elements in the preimage and multiply all of the things in the preimage, $b_j = \prod_{i \in f^{-1} j} a_i$; if it's empty it's the unit.

[picture]

A lemma, not really hard, is to check that $(f \circ g)_* = f_* \circ g_*$, and that's exactly where commutativity is needed.

You can see that if I'm doing something in the next step, if I compose things together I don't know how I'm multiplying things in the end. So anyway this is actually a functor from finite sets to commutative differential graded algebras, the defining property is somehow that $A \otimes A \to A$ is a map of algebras when A is commutative.

Before I go to simplicial sets, you can extend this construction to all sets by colimits, the tricky part is the tensor product for an infinite set, which is the colimit over finite things. So $A^{\otimes X}$ is the colimit over finite subsets of $A^{\otimes K}$. Now we extend it to simplicial sets. We want to model spaces by simplicial sets, let's take X_0, X_1, X_2, and so on be a simplicial set, that means that we are thinking of this as a space where X_0 are vertices, X_1 are edges glued on the vertices, X_2 are triangles glued on the edges, and so on. It's like a simplicial complex but we keep track of degeneracies everywhere.

Let me remind you about geometric realization, for a simplicial set, you have $\amalg X_n \times \Delta^n$ modulo the relation that $(d_i x, t) \sim (x, \epsilon^i t)$ and $(s_j x, t) \sim (x, \eta^j t)$, where ϵ^i and η_j are the comsimplicial structure on $\Delta^\bullet$.

The example to start with is X_0 a point, X_1 having a single non-degenerate one cell, and that's it.

Every time you have something like that, you only need to take care of the non-degenerate simplices. But to remember the homotopy and do some bookkeeping, multiplying two simplicial sets, you should use the degeneracies to remember you have small things in higher dimensions.

The best way to remember this thing is that there is a canonical map from $\Delta^n \times X_n \to |X_\bullet|$ given by the quotient map, $\bar{t}, \sigma \mapsto (\sigma, \bar{t})$. My convention for the simplex is that it's elements $0 \le t_1 \le \cdots \le t_n \le 1$.

Let's do the interval, which has I_0 equal to two point, I_1 having one non-degenerate simplex (and two degenerate things from the two points) and in higher dimension I_n will have $n+2$ things, all degenerate. Two of them correspond of the degeneracies of the two points and n of them for the degeneracies of the 1-simplex.

Basically I'll write I_n as $\{\underline{0}, \ldots \underline{n+1}\}$, where the underline corresponds to the two degeneracies of the vertices.

For this interval, the degeneracies occur when you discard all but one (or none) of the t_i points and the face maps are obtained by putting adjacent points together.

What do you get if you start with the standard model for the simplicial circle? Take the model S^1_n, where you identify these endpoints, and that is the same thing as writing $\{\underline{0}, 1, \ldots, n\}$, and you have the same structure $\varphi(\bar{t}, i) = t_i$ if i is between 1 and n and $\varphi(\bar{t}, 0) = 0$. Now let me explain what I was doing with my algebra. If X was a simplicial set, we get a simplicial commutative differential graded algebra $A^{\otimes X_0}$, $A^{\otimes X_1}$, and so on, with faces and degeneracies. In particular you have a simplicial chain complex or vector space and make a chain complex out of that in the standard way, define $CH_{X_\bullet}(A)$ as $\bigoplus_{n\ge 0} A^{\otimes X_n}$ where $A^{\otimes X_n}$ is in homological grading n and the differential d is given by the alternating sum $(-1)^i (d_i)_*$, where the d_i are the face maps $X_n \to X_{n-1}$.

You can extend to chain complexes using the inner differential in the same way, $A^{\otimes X_n}$ means the shifted complex $A^{\otimes X_n}[-n]$, the shifted complex, and you need to add a shift to the differential that is internal because of that shift.

Now you can check as an exercise that $CH_{S^1_\bullet}(A) = C(A)$ the standard Hochschild chain complex. The exercise is really easy because $CH_{S^1_\bullet}(A)$ is $A^{\otimes S^1_n}$ which is isomorphic to $A^{\otimes(n+1)}$, and identifying t_i and t_{i+1} in the differential, you get a term that looks like $a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$. Then the final one is the last multiplication for the weird term at the end.

The theory tells you that if you have used a different model of S^1 then do you get a related chain complex? If not, that's a little weird.

Proposition 4.1. *If $X_\bullet \to Y_\bullet$ is a map of simplicial sets which is a weak equivalence (it's enough to be a homology equivalence) then $CH_{X_\bullet}(A) \xrightarrow{f_*} CH_{Y_\bullet}(A)$ is a quasi-isomorphism. In other words, any simplicial set model of a space gives a canonical quasi-isomorphism. You can always compare two of them because you have a canonical map, $X_\bullet \to \Delta_\bullet(|X_\bullet|)$ (where this last is defined as $\mathrm{Maps}(\Delta^\bullet, |X_\bullet|)$).*

Since $A^{\otimes X_\bullet}$ is a simplicial commutative differential graded algebra, then the chain complex $CH_{X_\bullet}(A)$ is a commutative differential graded algebra with algebra structure given by the shuffle product.

So for $CH_{X_p}(A) \otimes CH_{X_q})(A) \to CH_{X_{p+q}}(A)$ you get something which factors through $CH_{X_{p+q}}(A \otimes A)$ (which then maps by multiplication to the desired target, and for that we take

$$x \otimes y \mapsto \sum (s_{\alpha_1})_* \circ \cdots \circ (s_{\alpha_q})_*(x) \otimes (s_{v_1})_* \circ \cdots \circ (s_{v_p})_*(y).$$

where the sum is over p, q-shuffles, where you premute v_1 through v_p and α_1 through α_q which maintains the relative order of each subset

I just want to say, tomorrow examples of higher dimensional things than the circle and give some other characterizations.

5. December 20: Sasha Voronov: Higher Categories and TQFTs II

Yesterday we were talking about closed topological quantum field theories in dimension 2. We actually touched upon the open-closed case, or actually just the open case, it was given as an exercise. But today I wanted to look at a mixture of those, of the open-closed two dimensional quantum field theory.

By the way, there is a good tradition of adding some letters in front of TQFT, you can talk about OCTQFT, ETQFT, CohFT, CFT, maybe T is an another addition, Q is also often dropped. We will be talking about OCTQFT, and here I'd like to generalize our setup to the following. Let me summarize about closed, I gave it to you as an exercise, that closed 2d-TQFTs are functors, on the category of cobordisms, monoidal functors, 2-dimensional cobordisms between 1-dimensional manifolds, to vector spaces.

For the open-closed, you want to modify the bordism category slightly, and you consider the category which I call $Bord_2$ which is better called $Bord_{0,1,2}$, and its objects are oriented compact one-dimensional manifolds, possibly with boundary. Morphisms between Y_- and Y_+ are diffeomorphism classes of two-dimensional cobordisms between Y_- and Y_+. These cobordisms may have corners.

Moreover, the morphisms from Y_- to Y_+ will also have other data: each boundary component—oh wait. I'm doing some dirty trick in including some data on the vector side on the objects of the category. Let me change the objects, they are oriented compact 1-manifolds, with boundary components labelled by points in a set $\mathcal{B}_0$. This is the set of boundary data of our open-closed field theory.

So say Y_- is an interval, this is a 1-manifold with boundary. Then Y_+ is like this [picture]. A morphism could look like this [picture]. For morphisms there is a condition that the boundary, manifolds with corners have boundary, take the boundary in the topological sense, and then you see several components, the boundary consists of Y_- and Y_+ and some other pieces. This could be thought of as the worldsheet of a propagating string. At some point it closed up and became a closed string. It opened again and then closed up and kept moving. If part of the initial boundary had boundary, it also has its worldpath. The boundary has a worldpath, and this is the other components of the boundary. The string acquired a boundary and traced a worldpath. The extra boundary that we get is called Y_c (for constrained boundary) and sometimes (amazingly enough, since they contradict each other), free boundary. So I should have constrained boundary labeled with elements of $\mathcal{B}_0$ in a way compatible with labelling on the boundary of objects.

So part of this is the "purely open sector" that we looked at yesterday [pictures]. This is just a planar sheet and this is another example.

Definition 5.1. A 2-dimensional open-closed topological quantum field theory is a monoidal functor from $\mathrm{Bord}_2^{\mathcal{B}_0}$ to vector spaces.

Another thing that I hould have said is that all of this is considered up to diffeomorphism.

As a consequence of the definition, both the cylinder and the strip give rise to the identity map, which was an assumption last time.

The question is, what do we have, how can we describe this possibly algebraically. Is there an analogue of the folklore theorem classifying 2-d closed field theories as Frobenius algebras.

Let's take a closer look at what's going on. If you have such a functor, there is only one connected closed one dimensional manifold, which corresponds to a vector space V. A union of several circles would give rise to $V^{\otimes n}$. What about the interval, we have only labelled intervals. Corresponding to this is a vector space $\mathcal{O}_{ab}$, using the orientation of the interval. The diagram here [picture] tells me that there is a product $\mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \xrightarrow{|\Sigma\rangle} \mathcal{O}_{ac}$. If you sketch a similar picture, part of the story is a closed field theory, the surfaces with no corners, oriented surfaces with just boundary, you get the closed topological field theory. So this is a commutative Frobenius algebra. There might be something else coming from the union of the open and closed sectors.

You see that the product is associative. If you also add units by sketching pictures, you can talk about a category $\mathcal{B}$, a $\mathbf{k}$-linear category $\mathcal{B}$ whose objects are $\mathcal{B}_0$ and whose morphisms $\mathrm{Mor}(a,b) = \mathcal{O}_{ab}$.

So you get this commutative Frobenius algebra plus the $\mathbf{k}$-linear category, and then there should be some extra conditions. What about a complete description that would say that this theory is equivalent to a certain algebraic object.

The complete description is given by a theorem of Moore and Segal in 2006, and it says the following.

Theorem 5.1. *A two-dimensional open-closed TQFT is equivalent to the following data:*

(1) *V a commutative Frobenius algebra*
(2) (a) *a collection $\mathcal{O}_{ab}$ of vector spaces for each a, b in $\mathcal{B}_0$ with an associative composition law*
    (b) *A non-degenerate trace $\theta_a : \mathcal{O}_{aa} \to \mathbf{k}$ which makes $\mathcal{O}_{aa}$ into a Frobenius algebra. Such a map, as before, defines a bilinear form if you compose*

it with the product, and this gives a perfect pairing on $\mathcal{O}_a a$ or an identification with its linear dual.

(c) *The compositions $\mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{aa} \xrightarrow{\theta_a} \mathbf{k}$ is also a perfect pairing. Moreover, $\theta_a(uv) = \theta_b(vu)$*

(3) *A linear map $\iota_a : V \to \mathcal{O}_{aa}$ and a linear map $\iota^a : \mathcal{O}_{aa} \to V$ satisfying*
   (a) *ι_a is an algebra homomorphism.*
   (b) *ι_a is central: $\iota_a(u)v = v\iota_b(u)$, and*
   (c) *ι_a and ι^a are adjoint:*

$$\theta_V(\iota^a(u)v) = \theta_a(u\iota_a(v))$$

   (d) *(Cardy condition) If we define π^a_b from $\mathcal{O}_{aa} \to \mathcal{O}_{bb}$ as $\sum_\mu e^\mu u e_\mu$ for a basis e_μ in $\mathcal{O}_{ab}$ and a dual basis e^μ for $\mathcal{O}_{ba}$, then this is $\iota_b \circ \iota^a$.*

[pictures]
Next time around I'll move to higher categories.

6. Andrei Căldăraru: Algebraic structures on Hochschild invariants of algebraic varieties and dg-categories II

It would be good to integrate the lectures, so I want to spend the first quarter of my lecture connecting what I'm doing to what Sasha was doing. I want to give a naive approach that shows how Hochschild homology could appear. Sasha has explained that an open closed TFT is a functor from the category of open-closed cobordisms to vector spaces. These are exactly the topological surfaces with boundary that Sasha was talking about. There is a lot of data so let me restrict to a couple of pieces of interest. I could forget the open part, looking only at manifolds without corners, and what I get from this, I'm looking at a subcategory of cobordisms, is a closed TFT. Or I could forget the closed part to open TFTs. An open-closed theory could give you just an open part or just a closed part. These open TFTs are symmetric Frobenius algebras while the closed ones are commutative Frobenius algebras.

A fundamental question you could ask about is whether there's a left adjoint to the forgetful functor to open TFTs. If someone gives you a purely open field theory, is there a universal open-closed field theory containing this open theory as its open part. If there is such a left adjoint (I'll tell you there is one), you could ask, what is the natural closed string sector associated to an open string sector. The answer is that the canonical closed string sector associated to an open TFT (and let me call the open TFT A) is, I need to give you a commutative Frobenius algebra, which is $C := A/[A, A]$, and the pairing is given by the formula $(a, b) = Tr_a m_b$ which is the trace of the operator $A \to A$ which sends x to axb. You should convince yourself that this does not change if you replace x by a multiple of a commutator. It's commutative because you've modded out by commutators. This is a theorem, I'm not claiming I've proved it. You could convince yourself that with this multiplication and pairing, this data together with A satisfies all the axioms that Sasha gave.

And what is $A/[A, A]$? It's the zeroth Hochschild homology.

There is something strange here. Nothing about C cares about the pairing of the open string sector. The open field theory had to have a trace, but that's not apparent. Also, we have nothing dg, and we only ended up with HH_0. What happened to higher Hochschild? If you really start doing things dg, you want to replace open-closed cobordisms with a dg version of it itself. We had some surfaces and we took homology of those. An object like this [picture] was discrete. If you put a conformal structure on your pair of pants, you get a moduli space of these things, then you'll get $\mathcal{M}_{g,n}$, and taking chains on that makes this into a dg object. This is Costello's theory for TCFTs. You turn the topological one to one which has some topology and some conformal structure.

I wanted to say, relative to Junwu's talk, that there is a natural theory of Hochschild homology for A_∞ algebras. We had our multiplication m_2, and he said there were these higher multiplications. Now when we were describing the Hochschild chain complex, we had like $a_0|\cdots|a_n$ and to apply b we collapsed two consecutive ones and then also brought the last one to the front: $m_2(a_0,a_1)|\cdots|a_n \pm \cdots \pm m_2(a_n,a_0)|a_1|\cdots|a_{n-1}$. If we have m_3 we have the same thing, a b_3 which does things like $m_3(a_0,a_1,a_2)|\cdots|a_n \pm \cdots \pm m_3(a_{n-1},a_n,a_0)|\cdots|a_{n-2}$.

Gregory explained that the way you do this is collapse things on the circle using a multiplication, if you have m_3 you collapse consecutive 3 points, if you had 4 points you'd do that too. If you have m_1, you do the same thing, apply to one thing. In an A_∞ algebra, m_2 does not need to be associative, but you might be worried because b_2 being a differential relied heavily on m_2 being associative. The total differential $b^2 = 0$ where $b = b_1 + b_2 + \cdots$, and in fact this is equivalent to the $\mathcal{A}_\infty$ relations.

I want to explain today, getting back to the material, I gave Hochschild homology and cohomology for a space X, say a smooth variety, we defined this as $HH^*(X) = \operatorname{Ext}^*_{X\times X}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$ or $HH_*(X) = R\Gamma(X, L\Delta^*\mathcal{O}_\Delta)$. But is this computable? I want to give you a very nice theorem, and one I'll actually try to prove, and before I state the theorem let me explain a little more. But let me use a simple adjunction. If I use Hochschild cohomology, this is $R\operatorname{Hom}_{D(X\times X)}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X)$. But now there's a fundamental result that lower star is adjoint to upper star, so this is the same (in the derived category of X if you want) to $R\operatorname{Hom}_{DX}(L\Delta^*\Delta_*\mathcal{O}_X,\mathcal{O}_X)$, which, $L\Delta^*\Delta_*\mathcal{O}_X$ is the same as $L\Delta^*\mathcal{O}_\Delta$, but now I can notice a strong similarity to the sections for Hochschild homology, which is $R\operatorname{Hom}_{D(X)}(\mathcal{O}_X, L\Delta^*\mathcal{O}_\Delta)$.

So these are almost dual. But what is the gadget $L\Delta^*\mathcal{O}_\Delta$? For $X = \operatorname{Spec}\ A$, then this is $(C_*(A),b)$. Find a resolution as an $A-A$-bimodule, then base change back to A. That's exactly what Hochschild was.

So conceptually, this is the scheme version of the Hochschild chain complex. But can we understand this reasonably well? Here's the main theorem, which has many authors.

Theorem 6.1. *(Swan, Yekutieli, Kontsevich, Kapranov) If X is smooth and the characteristic of the ground field is 0 or greater than the dimension of X, then $L\Delta^*\mathcal{O}_\Delta \cong \bigoplus \Omega^i_X[i]$ as objects in $D^b(X)$.*

How do I read this? Saying these are isomorphic in the derived category says that they are quasi-isomorphic. I can calculate the cohomology sheaves, which are the sheaves of differential forms. In particular, $H^{-k}(L\Delta^*\mathcal{O}_\Delta) = \Omega^k_X$. I've said HKR is the affine version of this, which says that the spaces of cohomology was the space of forms, but we have an explicit map which gives an isomorphism.

This theorem then tells me the Hochschild homology and cohomology as vector spaces. This implies, automatically, that

$$HH^*(X) \cong \bigoplus_{p+q=*} H^p(X, \wedge^q TX)$$

and

$$HH_*(X) \cong \bigoplus_{q-p=*} H^p(X, \Omega^q_X).$$

How do I do this? I want to compute for $HH_*(X)$, maps from $\mathcal{O}_X$ to this chain complex. If I'm looking at HH_0, then I'm computing maps from $\mathcal{O}_X$ to $\mathcal{O}_X$ with no shift, that's in the $p = q = 0$, or I could look at $p = q = 1$ and get something that lands in Ω^1 and look in H^1 of that. If I want to look at HH_1 then I need to shift my complex but everything else is unchanged. Does this make sense? Any questions?

In particular, it's nice because Hochschild homology can be calculated from knowing the Hodge diamond of a variety, if I write the Hodge diamond with $h^{p,q}$ and so on, then the dimension of HH_0 is the sum of the numbers $p = q$, the dimension HH_n is $h^{n,0}$ and HH_{-n} has dimension $h^{0,n}$. So this lives in degrees from $-n$ to n where n is the dimension of the variety. This has a very nice consequence, I'll spend zero time on this but it's important. If X is compact, then the numbers on one side are the same as on the other.

Theorem 6.2. *If X is compact, there's a natural duality induced by the Mukai pairing saying that $HH_i(X) \cong HH_{-i}(X)^\vee$.*

To prove this, you first construct the Mukai pairing (using wedge and integrate with a correction from the Todd genus) and then show it's nondegenerate. It's very similar to Serre or Poincaré duality. This is mirror to the Poincaré pairing. Mirror symmetry should rotate the Hodge diamond on the side and get the Betti numbers.

The duality here is totally natural without any choices or anything like that.

Let me outline a proof. We want to construct a map $L\Delta^*\mathcal{O}_X$ to $\bigoplus \Omega^i[i]$. A map in the derived category is a quasi-isomorphism is a local statement, that the induced maps in homology are isomorphisms, that's something to check locally. Checking locally because the diagonal is a local complete intersection, a smooth variety inside another, this can be calculated using a Koszul resolution. So the hard thing is constructing the map.

By adjunction again, I want a map from $\mathcal{O}_\Delta \to \bigoplus \Delta_*\Omega^i[i]$. This is now a map on $X \times X$, not on X.

But a map to a direct sum is a sequence of maps to each individual term. The first map is $\mathcal{O}_\Delta \to \mathcal{O}_\Delta$. When $i = 1$, this is the most interesting map. I want a map $\mathcal{O}_\Delta \to \Omega^1_\Delta[1]$. So think about this, such a map in the derived category of $X \times X$ means nothing but an extension, an Ext^1, which will look like

$$0 \to \Omega^1_\Delta \to ? \to \mathcal{O}_\Delta \to 0$$

Does anyone know what to put in the middle? The first jets, so $\mathcal{O}_{\Delta^{(1)}}$ is $\mathcal{O}_{X\times X}/\mathcal{I}^2$ where $\mathcal{I} = \mathcal{I}_\Delta$. This extension is called the universal Atiyah class, and I take that map. Now how do I go from $i = 1$ to $i = 2$? Now I need a map $\mathcal{O}_\Delta$ to $\Omega^2_\Delta[2]$. I break this up into a few steps. I start with the map I already had $\Omega^1_\Delta[1]$, and then I go from there to $\Omega^{\otimes 2}_\Delta[2]$ and from there to $\Omega^2_\Delta[2]$ via antisymmetrization, from the second tensor to the second exterior power.

The only question is, I take $\mathrm{At}^1 \otimes \pi_1^*\mathrm{id}_{\Omega^1[1]}$. I get a short exact sequence from above by tensoring:

$$0 \to \Omega^{\otimes 2}_\Delta \to \pi_1^*\Omega^1 \otimes \mathcal{O}_{\Delta[unintelligible]} \to \Omega^1_\Delta$$

and then I do the same thing moving forward, just repeat. I take my map $\sum \frac{1}{i!} \mathrm{At}^i$ (and you see this interesting factor we saw in HKR) and now you're basically done, the rest is a local check which is easy with a Koszul resolution.

By the way, I want to say one more thing related to the proof but not to Hochschild homology. The same proof gives the following theorem

Theorem 6.3. *(Annkin, C.) If $i : X \hookrightarrow Y$ is a closed embedding of smooth things, then*

(1) $H^{-k}(i^* i_* \mathcal{O}_X) \cong \wedge^k N^\vee_{X/Y}$
(2) *$i^* i_* \mathcal{O}_X$ is formal, is quasi-isomorphic to its homology computed in the first item, if and only if the conormal bundle extends to $X^{(1)}$*

I have five more minutes. I want to give a preview of what comes next. This proof I gave is not nice. Kontsevich and Kapranov had a better conceptual understanding. The next ideas are that we can think of $L\Delta^*\mathcal{O}_\Delta$ as $\mathcal{O}_{LX}$, the structure sheaf of some geometric object. The $L\Delta^*\mathcal{O}_\Delta$ is a commutative dga, this is the functions on a generalized space, we should think of this as the fiber product of the diagonal with itself over $X \times X$, and using an analogy with topology, is the same as the maps from S^1 to X. This is closely related to what Gregory has been explaining. The maps from S^1 to X should have fibers groups. If you want to understand a Lie group, locally they are isomorphic to their Lie algebras. The dg Lie algebra is the shifted tangent bundle $TX[-1]$. Now what we are proving is, the standard theorem from Lie theory says that the exponential gives a (formal) isomorphism between the group and the algebra, and I think that I have this family of Lie groups LX with the family of corresponding Lie algebras $\mathrm{Tot}\, TX[-1]$, and the map is the exponential map, and when you see this thing, well, this explains the $\frac{1}{i!}$. This will be our next goal, to understand the next level of similarity. That's tomorrow.

7. Junwu Tu: Homotopy L_∞ spaces and its applications II

I'll continue from last time. I defined what is an A_∞ algebra. It's a vector space with the m_k multiplications, $k \geq 1$, still over a field $\mathbf{k}$, and the definition was complicated of course, I gave a construction that makes the definition easier to understand, this is the bar construction. It works in general but I'll assume dim $A < \infty$. It looks like an algebraic trick. So I start with $m_k : A^{\otimes k} \to A$, and then I do a shift, I look at $m_k : A[1]^{\otimes k} \to A[1]$, the same map but up to sign. Then I dualize, get $m_k^\vee : (A[1])^\vee \to (A[1]^{\otimes k})^\vee \cong (A[1]^\vee)^{\otimes k}$, and so we know that the dual of a shifted graded vector space, $A[1]^\vee = A^\vee[-1]$. Then we have this for every $k \geq 1$. Now I put these maps together and get,

$$\prod_{k=1}^{\infty} m_k^\vee : A^\vee[-1] \to \prod_{k=0}^{\infty} (A^\vee[-1])^{\otimes k}$$

The right hand side is the completed tensor algebra $\widehat{T}(A^\vee[-1])$, completed because this is the direct product not the direct sum. If this tensor was the symmetric algebra, then we thing $\hat{\mathbb{S}}(V^\vee)$ is formal functions $\mathcal{O}_{V,0}$, so this is formal non-commutative functions, and the space that we're looking at is $A[1]$, formal functions (at 0) of this graded vector space.

It's a general property that if you have a map from $V^\vee$ to $\hat{\mathbb{S}}(V^\vee) \to \hat{\mathbb{S}}(V^\vee)$. So an $\tilde{m}$ gives rise to a derivation here such that $[m, m]$ vanishes.

You can also give this for morphisms, $f : A \to B$ is an A_∞ homomorphism, f is by definition given by multilinear maps, I still denote by the shifted one $f_k : A[1]^{\otimes k} \to B[1]$, and I dualize to get $f_k^\vee : (B(1))^\vee \to A[1]^{\vee \otimes k}$ and put it together $f_k^\vee : (B[1])^\vee \to \mathcal{O}^{\mathrm{NC}}_{A[1]}$ and this extends to $\tilde{f} : \mathcal{O}^{\mathrm{NC}}_{B[1]} \to \mathcal{O}^{\mathrm{NC}}_{A[1]}$, the same picture as in the commutative space. Now we are doing commutative geometry. The induced map $\tilde{f} : \mathcal{O}^{\mathrm{NC}}_{B[1]} \to \mathcal{O}^{\mathrm{NC}}_{A[1]}$, the condition of being an A_∞ homomorphism is that

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{NC}}_{B[1]} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{m}_B"] & \mathcal{O}^{\mathrm{NC}}_{A[1]} \arrow[d, "\tilde{m}_A"] \\
\mathcal{O}^{\mathrm{NC}}_{B[1]} \arrow[r, "\tilde{f}"'] & \mathcal{O}^{\mathrm{NC}}_{A[1]}
\end{tikzcd}$$

commutes.

So this has lots of applications. Look at the Hochschild chains, $C_*(A)$, then $C^*(A)$ is derivations of $\mathcal{O}^{NC}_{A[1]}$ with differential given by bracketing with the vector field $[\tilde{m}_A, \quad]$. In the finite dimensional case, $C_*(A)^\vee$ is the space of noncommutative one-forms, and the vector field acts by Lie derivative, and this is $C_*(A)^\vee \cong (\Omega^{\mathrm{NC}}_{A[1]}, L_{\tilde{m}_A})$.

What I want to talk about is not A_∞ algebra but L_∞ algebra. I'll take a minute to define those.

I could have written down a complicated one, $\ell_K : \wedge^k g \to g$ of degree $2-k$ and this should satisfy a complicated equation, this is Q, a derivation of $\hat{\mathbb{S}}(g^\vee[-1])$ such that $[Q, Q] = 0$, and the degree of Q is 1. You can define L_∞ homomorphisms in the same way.

So let me make a table

vector spaces	L_∞-algebras
vector bundle (locally free module over the functions $\mathcal{O}_M$)	L_∞-algebra bundle (again locally free over $\mathcal{O}_M$): $\widehat{\mathbb{S}}(g^\vee[-1])$ and Q a derivation, $\mathcal{O}_M$-linear
vector bundle with flat connection	L_∞ spaces
vector bundle with flat superconnections (Ω^*_M-modules)	Costello's L_∞ spaces

Let me give the definition

Definition 7.1. An L_∞-space is a triple (M, g, D) such that

(1) g is an L_∞ algebra bundle of the form

$$g = T_M[-1] \oplus g^2 \oplus \cdots \oplus g^d$$

(positively graded), and we want to look at the Chevalley–Eilenberg complex $C^*_g = (\widehat{\mathbb{S}}(g^\vee[-1], Q)$, so that $\widehat{\mathbb{S}}(g^\vee[-1]) \cong \widehat{\mathbb{S}}(\Omega^1_M) \otimes \widehat{\mathbb{S}}(g^{\geq 2\vee}[-1])$ with this first factor in degree zero.

(2) D is a connection. This is not flat in each degree, but the right notion is not a flat connection on the Lie algebra itself but on the Chevalley–Eilenberg algebra. So $D : C^*g \to \Omega^1_M \otimes C^*_g$, a flat ($D^2 = 0$), and there are lots of properties:
   - the internal degree of D is zero, so it keeps the symmetric degree and only has form degree 1

- D is a derivation, I'm not assuming this is $\mathcal{O}_M$-linear, only $\mathbb{R}$-linear, and it acts on functions $\mathcal{O}_M$ by de Rham d
- $[D,Q]=0$
- Restricted to the 1-form part $D:\Omega^1_M\to\Omega^1_M\otimes\mathcal{O}_M$ sends α to $\alpha\otimes 1$.

The definition is not important. The two examples, very simple, will be what's important, you can interpret them in terms of L_∞ algebras, and that'll give you an advantage.

Example 7.1. Consider the case $g=T_V[-1]$, the tangent bundle of V shifted. So V is an open subset of $\mathbb{R}^n$. If you work out the degree, ℓ_k, when you take exterior powers, this is $\operatorname{Sym}^k T_V\to$ degree 2, which is zero, since the degree of ℓ_k is $2-k$. So we start with a trivial L_∞-algebra bundle. What is D? If we write the Chevalley–Eilenberg complex down, we get $C^*g=\widehat{\mathbb{S}}(\Omega^1_M)$, all in degree 0. The connection part, $\widehat{S}(\Omega^1)\to\Omega^1\otimes\widehat{S}(\Omega^1)$. How do we see what should be this D? I have this D, it's a derivation of the algebra structure, and I know it's $\mathbb{R}$-linear, so this is uniquely determined by what it does on the generators, but the generators as an $\mathbb{R}$-algebra, and this is generated by $\mathcal{O}_V\oplus\Omega^1$. Now by the axioms that I gave, $D|_{\mathcal{O}_V}$ is d_{dR}, fixed by the definition. All I have to do is say what it has to do on the one-form part, and there one part is fixed, we have $\Omega^1\mapsto\Omega^1\otimes\mathcal{O}\oplus\Omega^1\otimes\Omega^1\oplus\Omega^1\otimes\operatorname{Sym}^2\Omega^1\oplus\cdots$

So I put for the first part τ and ∇ a connection for the first component and zero elsewhere. So this is $D=\tau+\nabla$.

We can verify everything in local coordinates. If I fix my chart, then $T_M=\langle\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x^n}\rangle$ and then I have the dual frame $\Omega_M=\langle\xi_1,\dots,\xi_n\rangle$, and then I'll make another copy $\Omega_M=\langle dx_1,\dots,dx_n\rangle$. So very concretely, $\widehat{\mathbb{S}}_{\mathcal{O}_V}(\Omega^1)=\mathcal{O}_V[[\xi_1,\dots,\xi_n]]$, and in this way, $\tau=dx_i\frac{\partial}{\partial\xi_i}$ and ∇ acts by de Rham, $\sum dx_i\frac{\partial}{\partial x_i}$.

You can check that $D^2=0$, this is the same as $[D,D]$, there are no ξ in one or x in the other so you get zero.

This should be thought of as a derived scheme. When you only have one tangent, $g=T_M[-1]$, by this construction we can calculate $\ker D$, and that's just $\mathcal{O}_V$, and this is a very explicit computation. This is how you connect this space to a commutative space.

Example 7.2. For $V\subset\mathbb{R}^n$, I can define $g=T_V[-1]\oplus\mathbb{R}^m[-2]$. Here the L_∞ structure does not have to be trivial, ℓ_k is a map from $\mathbb{S}^kT_V\to\mathbb{R}^m$, this is what ℓ_k could be, and if you put anything in degree two, you'll get 0. The good thing is that L_∞ identity uses two ℓ_k and so any bundle map satisfies the L_∞ condition. We don't have to check the complicated axioms.

So again I choose coordinates $x_1,\dots,x_n$, and $f_1,\dots,f_m$ for $\mathbb{R}^m$, a local frame for the trivial bundle. The Chevalley–Eilenberg looks like $\widehat{\mathbb{S}}(\Omega^1_V)\otimes\wedge\langle f_1^\vee,\dots,f_m^\vee\rangle$, with the first factor in degree 0 and the generators on the second factor in degree -1. This is C^*g, and I have this Q coming from the ℓ_k, and again I want to put $D=\tau+\nabla$ as before. There is a fixed part τ, and then there is a connection on Ω^1_V given by the local coordinates, and you can verify, this is a computation that you can do, so $D^2=0$, there is nothing changed here, but there is a compatibility $[Q,D]=0$ which happens if and only if, now we have some restriction on the maps, we need

$$\nabla_{\frac{\partial}{\partial x_i}}\ell_k(\frac{\partial}{\partial x_{i_1}},\dots,\frac{\partial}{\partial x_{i_k}})=\ell_{k+1}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_{i_1}},\dots,\frac{\partial}{\partial x_{i_k}})$$

which means that $\{\ell_k\}$ is determined by ℓ_0, which is just a section.

So whenever you have an L_∞ space with just the tangent bundle in degree 1, it's isomorphic to this first example, and if you have [unintelligible]then it's isomorphic to the second example.

$$(\Omega^*_M \otimes C^*_g, D + Q)$$

is isomorphic to the derived critical locus

$$0 \to \wedge^{\mathrm{top}}(\mathbb{R}^m)^\vee \xrightarrow{\lrcorner \ell_0} \cdots \xrightarrow{\lrcorner \ell_0} (\mathbb{R}^m)^\vee \xrightarrow{\lrcorner \ell_0} \mathcal{O}_M$$

8. Grégory Ginot: Higher Hochschild homology and factorization algebras II

Yesterday we have seen that if we have $X = |X_\bullet|$ with $X_\bullet$ a simplicial set, we can define the higher Hochschild homology for X, which was

$$CH_{X_\bullet}(A) = \left(\bigoplus_{n\geq 0} A^{\otimes X_n}, b\right)$$

Example 8.1. We have a nice simplicial model for the torus from S^1, $T_n = S^1_n \times S^1_n = \{\underline{0}, \dots, n\} \times \{\underline{0}, \dots, n\}$, and you can think of this as being like a kind of matrix. Okay, so it's, obviously what I've been doing as taking what Gabriel calls the stupid model for the torus, where we have been identifying the opposite sides of a square, and what we have been doing is, $\varphi : I_n \times \Delta^n \to S^1 \times S^1 = |T_n|$, so

$$\varphi((i,j); \bar{t} = 0 \leq t_1 \leq \cdots \leq t_n \leq 1) = (t_i, t_j)$$

where $t_0 = 0$, my marked point.

[pictures]

Then the rule on $CH_T(A)$, so $A^{\otimes T_n}$, this is $A^{\otimes (n+1)^2}$, for every point I put an element a_{ij}, on this grid on the torus square.

What is the simplicial structure? I get $d_i : T_n \to T_{n-1}$ when $t_i = t_{i+1}$. [pictures]

So what's really going on in the end, here's my torus [pictures], I multiply the rows i and $i+1$ and multiply the columns i and $i+1$. In the middle I multiply four guys together.

Example 8.2. For the two-sphere, take $S^1_\bullet \wedge S^1_\bullet$, so kill all the boundary, we have $S^2_n = \{\underline{0}, \dots, n^2\} = \{\underline{0}\} \amalg \{1, \dots n\}^2$.

What's the presentation? We have n^2 guys in the middle and then one point along the boundary. The simplicial map multiplies adjacent rows and columns together. You have nothing on the exterior in these lines. The only difference is going to happen when $i = 0$ or $i = n$. If $i = 0$, then the picture looks like this, where the first row and column is all multiplied into a_0, and similarly for $i = n$.

Now these two things look the same because of the similar presentation but since the differential is different they have different homologies. You can make a several genus surface. Let me make a comment for higher spheres. You have a model of $S^d_\bullet$ which is $S^1_\bullet \wedge \cdots \wedge S^1_\bullet$ so then S^d_n will be $\{0\} \amalg \{1, \dots, n\}^d$ and instead of a square you'll have a higher cube and in the middle you will just be multiplying all the rows and so on.

This is not the smallest model you can take for the sphere. This is a simplicial set describing S^d as $I^d/\partial I^d$. You have d many one-cells and so on.

There is a smaller one, given by $S^d = D^d/\partial D^d$. If you do that, then $S^{d,\text{small}}$, which has $S_n^{d,\text{small}} = \{\underline{0}\}$ for $0 \le n \le d-1$, and $S_d^{d,\text{small}} = \{\underline{0}, 1\}$, and in fact for $n \ge d$ it is $\{\underline{0}, 1, \dots, \binom{n}{d}\}$.

That's a much smaller model. It has some interest, it makes it much easier to compute the first cohomology group of that thing. What is the chain complex in this situation? We have $CH_{S_\bullet^{d,\text{small}}}(A)$ is the following:

$$A \leftarrow A \leftarrow \cdots \leftarrow A \leftarrow A \otimes A \leftarrow A^{\otimes d+2} \leftarrow \cdots$$

and so you get 0, the identity and back and forth. So I have A in degree zero and then no cohomology in degree 1 and so on. So $HH_{S_\bullet^{d,\text{small}}}(A)$ is A in degree 0, is 0 in degree $\le d-1$, and then Ω_A^1 in degree d. That computation is not hard to do in this setting.

So the remark is just that using this model it is easy to do a computation. That's what I wanted to say, and I think I wanted to go, let me say some properties of Higher Hochschild.

Theorem 8.1. *(G., Tradler, Zeinalian) $CH_{(\ )}(\ )$ from either* $\mathrm{Top} \times \mathrm{CDGA}$ *or* $\mathrm{sSet} \times \mathrm{CDGA}$ *to* CDGA *(switching between topological spaces and simplicial sets using singular and geometric realization) satisfies*

(1) *it is homotopy invariant with respect to maps of spaces and commutative differential graded algebras (meaning a quasi-isomorphism of algebras induces a quasi-isomorphism of the result) (homotopy invariance)*
(2) $CH_{pt}(A) \cong A$ *(as algebras) (dimension)*
(3) $CH_{\amalg X_i}(A) \cong \bigotimes CH_{X_i}(A)$ *(as algebras) (symmetric monoidal)*
(4) *This should be Mayer–Vietoris, if $X \subset Z$ is a subcomplex and you have $f : X \to Y$, then we can glue Y to Z by identifying, we have $Z \cup_X Y$, which is the pushout, and in that situation we have four algebras, in that situation, this gives you a map of algebras $CH_X(A) \to CH_Z(A)$ and $CH_X(A)$ to $CH_Y(A)$, and we have maps $CH_Y(A) \to CH_{Z \cup_X Y}(A) \leftarrow CH_Z(A)$, and in particular you have a canonical map,*
$$CH_Y(A) \otimes^{\mathbb{L}}_{CH_X(A)} CH_Z(A) \to CH_{Z \cup_X Y}(A)$$
is a quasi-isomorphism. That's the statement.
(5) *(this is really cool) Any functor* $\mathrm{Top} \times \mathrm{CDGA}$ *satisfying the above is naturally equivalent to the one I've given, this axiom determines the functor.*
(6) *There is a natural (weak) equivalence*
$$\mathrm{Maps}_{\mathrm{sSet}}(X_\bullet; \mathrm{Maps}_{\mathrm{CDGA}}(A, B)) \cong \mathrm{Maps}_{CDGA}(CH_{X_\bullet}(A), B).$$
This could have been taken as a definition of the functor.

In the case I described above, this is not just a pushout but a homotopy pushout, and what I want to say is that this result is true (and more interesting in some sense) for any homotopy pushout. Once you have a homotopy pushout, you get a cofibration, and then you get this canonical map, this is more general although they are implied by each other.

That derived tensor product is in the category of commutative differential graded algebras. This means that to compute that thing, you should resolve one of the two sides as a module which is free (well, cofibrant) over $CH_X(A)$.

For $S^2 = D^2 \cup_{S^1} D^2$ and what we're saying is that $CH_{S^2}(A) \cong CH_{D^2}(A) \otimes_{CH_{S^1}(A)} CH_{D^2}(A)$, so this is $A \otimes^{\mathbb{L}}_{CH_{S^1}(A)} A$, so if D^2_n is $I_n \times I_n$ and S^1 is ∂D^2. This is not the usual model, let me note that this is defferent than the S^1 that I was describing. So the model has $\{\underline{0}, \ldots, \underline{n+1}, \ldots, \underline{2n+2}, \ldots, \underline{3n+3}, \ldots, 4n+3\}$, and then the D^2_n guy is semifree essentially by definition because we can embed that boundary. But that's not the most effective way. Instead I could take $CH_{D^2_\bullet}(A)$, this is $CH_{\partial D^2}(A) \otimes A^{\otimes n^2}$, and with a differential which stays on the left and a part that interacts on the right ,and this is a semifree algebra over $CH_{\partial D^2_n}(A)$. That's not easy to compute.

If A was smooth, say $A = \mathrm{Sym}(V)$, then we have $CH_{S^1}(A) \cong \mathrm{Sym}_A(\Omega^1_A[1])$, and this is really $\mathrm{Sym}(V \oplus V[1])$, and now the question is, I know $CH_{S^2}(A) \cong A \otimes^{\mathbb{L}}_{\mathrm{Sym}(V \oplus V[1])} A$, and that's $\mathrm{Sym}(V) \otimes^{\mathbb{L}}_{\mathrm{Sym}(V \oplus V[1])} \mathrm{Sym}(V \oplus V[1] \oplus V[2], \partial : V[2] \to V[1])$ and that will look like $\mathrm{Sym}(V \oplus V[2]) \otimes^{\mathbb{L}}_{\mathrm{Sym}(V)} \mathrm{Sym}(V) \cong \mathrm{Sym}(V \oplus V[2])$.

Finding a semifree resolution will always be easy and that will do the semifree thing for you. Now that you know you have this result, you pick the convenient model and you know how to make this computation much easier.

I want to mention two statements and stop.

Corollary 8.1. *("Fubini formula") There is a canonical equivalence of commutative differential graded algebras between $CH_X(CH_Y(A)) \cong CH_{X \times Y}(A)$.*

Proof. Or $CH_{(- \times Y)}(A)$ satsifies the axioms with A replaced by $CH_Y(A)$. □

So for instance, $C_*(C_*(A)) \cong CH_{T^2}(A)$.

The other comment I wanted to say is that any map from X to Y tells you, if you have a group on X and you want the group to act on CH_X, this tells you what to do.

Let me state the HKR theorem.

Theorem 8.2. *(HKR) If X is a formal space (e.g. a sphere or a suspension or a Lie group or a Kähler manifold) then $A = (\mathrm{Sym}(V), \partial)$, then $CH_X(A) \cong (\mathrm{Sym}(V \otimes H_*(X)), d)$ as a commutative differential graded algebra, with, if $\partial(v) = \sum_{i \geq 0} v_{a_1} \bullet \cdots \bullet v_{a_i}$, and then if $\sigma \in H_k(X)$, then*

$$d(v \otimes \sigma) = \sum_{i \geq 0} (v_{a_1} \otimes \sigma^{(1)}) \bullet \cdots \bullet (v_{a_i} \otimes \sigma^{(i)})$$

where $\Delta^{(i)}(\sigma) = \sum \sigma^{(1)} \otimes \cdots \otimes \sigma^{(i)}$, where Δ is the coalgebra product on $H_*(X)$ dual to cup product.

9. Sangwook Lee: Mirror symmetry of Calabi–Yau categories

Today I will cover

- the Fukaya category,
- the pairing $\langle\rangle_{\mathrm{Fuk}}$,
- the pairing $\langle\rangle_{\text{Kapustin–Li}}$,
- ks
- $OC : HH_*(\mathrm{Fuk}(X)) \to QH_*(X)$
- $\eta : HH_*(MF(W)) \to Jac(W)$
- results

For me (M, ω) is a symplectic manifold.

Definition 9.1. The *Fukaya category* $Fuk(M)$ $\mathcal{A}$ is an A_∞-category (over a (Novikov) field Λ) with objects Lagrangian submanifolds, and $\hom_{\mathrm{Fuk}}(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \Lambda p$.

Recall that an A_∞ category consists of objects, morphism sets, and m_k maps

$$\hom_{\mathcal{A}}(A_1, A_2) \otimes \cdots \otimes \hom_{\mathcal{A}}(A_k, A_{k+1}) \to \hom_{\mathcal{A}}(A_1, A_{k+1})$$

satisfying Tu's A_∞ relation. The m_0 means that we assign an element m_0^A for each object A.

We should define the structure maps in the Fukaya category. If we have $L_1 \to L_2 \to \cdots \to L_{k+1}$, so we have $p_1, \ldots, p_k$, intersection points of L_i and L_{i+1}, then $m_k(p_1, \ldots, p_k)$ means that we count holomorphic $(k+1)$-gons with boundary on L_i and vertices on $p_1, \ldots, p_k$ and an output, so I evaluate on the output point, weighted by the symplectic area. I chose an almost-complex structure compatible with the symplectic form.

This will also be covered by Sasha tomorrow so I won't be very precise about this.

[some discussion about m_0^A]

Definition 9.2. A Lagrangian L is *weakly unobstructed* if there is $b \in \hom_{\mathrm{Fuk}}(L, L)$ such that $m_0^b \coloneqq m_0^L + m_1(b) + \cdots + m_k(b, \ldots, b) = W_L \mathrm{id}_L = W_L[L]$, and we should choose these things properly so that this kind of equation holds.

Definition 9.3. The pairing $\langle\rangle_{\mathrm{Fuk}}$, the strip in Sasha's talk, is a pairing

$$\hom_{\mathrm{Fuk}}(L, L') \otimes \hom_{\mathrm{Fuk}}(L', L) \xrightarrow{m_2} \hom_{\mathrm{Fuk}}(L, L) \xrightarrow{\mathrm{tr}} \Lambda,$$

and this trace is $\int_L$ under the identification (that I'll always take) of $\hom_{\mathrm{Fuk}}(L, L) \cong C^*(L)$.

So we read the coefficient of $[\mathrm{pt}_L]$.

Why do we care about the Fukaya category? This is one part of homological mirror symmetry.

Conjecture 9.1. (Kontsevich) Suppose M and M' are mirror Calabi–Yau manifolds. Then $D^\pi \operatorname{Fuk}(M) \cong D^b \operatorname{Coh}(M')$ and vice versa, as triangulated categories.

Don't worry about this notation, which we won't use today.

Beyond Calabi–Yau manifolds, the statement changes:

$$D^\pi \operatorname{Fuk}(M) \cong H^0 MF(W)$$

the category of matrix factorizations, where $W : (\mathbb{C}^*)^n \xrightarrow{\mathrm{hol}} \mathbb{C}$.

People often show this by showing A_∞ equivalences on generating categories of each side. So I also need to talk about matrix factorizations.

Definition 9.4. Let W be central in an associative algebra R, then a *matrix factorization* of W is the data $p_0 : P^0 \to P^1$ and $p_1 : P^1 \to P^0$, such that P^i is a free R-module and $p_0 \circ p_1 = W \mathrm{id}_{P^1}$ and $p_0 \circ p_1 = W \mathrm{id}_{P^0}$.

An easy fact is that $\mathrm{MF}(W)$ is a differential graded category, like chain complexes.

Today I'll talk about the case $R = \Lambda[x_1, \ldots, x_n]$, or a Laurent polynomial ring, and W is a polynomial with isolated singularities.

I think of Fuk as forming a sort of open TCFT maybe, this has holomorphic data. We also want to treat MF as a sort of TCFT. This is not all the data we need, but I will anyway talk about this kind of thing, I want to warn you that in a strict sense we will not get a Calabi–Yau category structure. I suppose that I have a matrix factorization, so I have $P^\bullet$, and I'll use the notation δ_P with $\delta_P^2 = W\mathrm{id}_P$. When we think of these as objects in the differential graded category, morphisms are R-linear maps.

So if I have a map φ, this is a matrix and δ_P is the same.

So I'll define a pairing $\langle , \rangle_{KL}$ of morphisms

$$\hom_{MF}(P^\bullet, Q^\bullet) \otimes \hom_{MF}(Q^\bullet, P^\bullet) \xrightarrow{m_2 = \circ} \hom_{MF}(P^\bullet, P^\bullet) \xrightarrow{\mathrm{tr}_{KL}} \Lambda.$$

Now if $P^\bullet$ was two pieces of degree n then this $\hom_{MF}(P^\bullet, P^\bullet)$ is then a $2n \times 2n$-size square matrix. Then

$$\mathrm{tr}_{KL}(\varphi) = \frac{1}{(2\pi i)^n n!} \oint_{\{|\partial_{x_i} W| = \epsilon|_{i=1,\dots,n}\}} \frac{\mathrm{tr}(\varphi \circ d\delta_{P^\bullet})^{\wedge n}}{\partial_{x_1} W \cdots \partial_{x_n} W}$$

So if I have $P^\bullet = R \oplus R$ with maps $(x+y)$ and (x^2+y) then $\delta_{P^\bullet} = \begin{pmatrix} & x+y \\ x^2+y & \end{pmatrix}$, and $d\delta_{P^\bullet} = \begin{pmatrix} & dx+dy \\ 2xdx+dy & \end{pmatrix}$ and we compute $(d\delta_{P^\bullet})$^2 in the usual way as $\left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right) dx \wedge dy$.

I can't make this strict on the chain level, but my trace should factor through Hochschild homology. I have a map $\hom_{Fuk}(L, L) \to \Lambda$ and $\hom_{MF}(P_L, P_L) \to \Lambda$, two different traces, and these, I request that they factor through Hochschild homology as chain maps:

$$\hom_{Fuk(L,L)} \to CH_*(\mathrm{Fuk}) \to \Lambda$$

and similarly for matrix factorizations. The map for matrix factorizations is the "boundary-bulk map" computed explicitly by Polishchuk–Vaintrob and Dickerhoff–Murfet.

Now I'd like to talk about closed string mirror symmetry which shows the isomorphism betwen $QH^*(X)$ and $\mathrm{Jac}(W)$. For X compact toric, So Fukaya–Oh–Ohta–Ono gave a geometric map $\bigoplus ks_L$ (for Kodaira–Spencer), then $\mathrm{Jac}(W_L)$ uses a counting of Maslov–index two holomorphic disks on Lagrangians. Their way to define the quantity is by considering these moduli spaces, this is free boundary, you pushforward under an evaluation map and they showed that this is given by $\boxed{\mathrm{a}}[L]$ so $ks_L(c) = \boxed{\mathrm{a}}$.

[pictures]

So the Hochschild chain complex of A_∞-categories are the same as in the algebra case. The Hochschild chain complex $(CH_*(\mathcal{A}), b)$ is

$$\bigoplus_{L_0,\dots,L_k} \hom(L_0, L_1) \otimes \cdots \hom(L_k - 1, L_k) \otimes \hom_{L_k, L_0}.$$

We always think of these tuples of morphisms, and

$$b(x_0, \dots, x_k) = x_0 \otimes \cdots \otimes m_*(x_{i+1}, \dots, x_j) \otimes \cdots \otimes x_k \pm m_*(x_{\ell+1}, \dots x_k, x_0, \dots x_i), x_{i+1}, \dots, x_\ell$$

and then $b^2 = 0$ due to the A_∞ equations.

The endomorphism space goes very naturally into this complex. The embedding map is actually a chain map.

The formula for the boundary-bulk map is given in a complicated way, but if we see Hochschild as a kind of bar, then this is nothing but the inclusion. In much of the literature, they call this map $HH_*(\mathrm{Fuk}(X)) \to QH^*(X)$, and here if you have a Hochschild homology element $p_0 \otimes \cdots \otimes p_k$, then the definition is that this tuple should go to the counting of [picture].

Okay, and so we have $HH_*(MF(W)) \to \mathrm{Jac}(W)$, and here if I have $\varphi_0 \otimes \cdots \otimes \varphi_k$, then the map takes this to $s\,\mathrm{tr} \circ HKR \circ \exp(-bD_W)$ applied to this element.

So $bD_f(\varphi_0 \otimes \cdots \otimes \varphi_k) = \varphi_0 \otimes \delta_{P^1} \otimes \varphi_1 \otimes \cdots \otimes \varphi_k \pm \cdots \pm \varphi_0 \otimes \cdots \otimes \varphi_k \otimes \delta_{P_0}$

and $HKR(\varphi_0 \otimes \cdots \otimes \varphi_k) = \frac{1}{k!}\varphi_0 d\varphi_1 \wedge \cdots \wedge \varphi_k$.

Finally we have a supertrace, the alternating trace.

I have a bunch of definitions and finally I can state our result.

Theorem 9.1. *(Cho–L.–Shin) Let X be compact toric Fano. Then*

$$\begin{tikzcd}
\hom_{\mathrm{Fuk}}(L,L) \arrow[r] \arrow[d, "F"'] & CH_*(\mathrm{Fuk}) \arrow[r] \arrow[d, "F_*"'] & QH^* \arrow[d, "ks"] \\
\hom_{MF(W)}(P_L,P_L) \arrow[r] & CH_*(MF) \arrow[r] & \mathrm{Jac}
\end{tikzcd}$$

commutes at the homology level, where F is the Cho–Hong–Lau map.

This allows us to say that F is an equivalence. So the result is that F intertwines the traces I have defined.

10. December 21: Sasha Voronov: Higher Categories and TQFTs III

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12. December 22: Junwu Tu: Homotopy L_∞ spaces and its applications III

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14. Sasha Voronov: Higher Categories and TQFTs IV

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18. Grégory Ginot: Higher Hochschild homology and factorization algebras IV