MPIM HIGHER STRUCTURES IN GEOMETRY AND PHYSICS (MARCH 2016)

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1. March 18: Julie Bergner: Comparing models for (∞, n)-categories

Thanks to the organizers for the invitation to speak and to be here for a couple of weeks of the program. The goal today is to look at some of the models for (∞, n) categories and talk about some ways that they can be compared. This won't be comprehensive and there are lots of other ways to do this, but there's only so much I can do in an hour.

So let me just start, if people are not so familiar with the idea of what these should be, with an intuitive sense, an (∞, n) category should have, it should by an ∞-category, so there should be objects, 1-morphisms between objects, 2-morphisms between 1-morphisms, and so on, and the n part says that at some point things should be weakly invertible. We want all k-morphisms to be weakly invertible for $k > n$.

Okay, so this is always going to be a bit inductive in nature. Let's start with the most basic case when $n = 0$, so we have $(\infty, 0)$-categories when all morphisms are invertible, so these are ∞-groupoids. I'll approach this by considering these to be topological spaces or Kan complexes.

The idea, in spaces for the moment (I'll get more simplicial as the talk goes on) is that the points are the objects, the paths are the 1-morphisms. We want everything to be weakly invertible, you can always run a path in the opposite direction. Then we can take homotopies between those paths, which gives 2-morphisms, and homotopies between homotopies, all of which are weakly invertible, you can run them all in the other direction.

I want to look at the homotopy theory of all the structures I'm looking at, this is the homotopy theory of topological spaces, this is where all of that started, but I want to treat these as having a *model category* structure. This is the model structure on topological spaces where the weak equivalences are the weak homotopy equivalences. Or we can take the equivalent structure on simplicial sets where the fibrant objects are Kan complexes.

How do we keep going from here? A general principle, and this is just a general principle with higher categories in general, is that an (∞, n)-category should be a category enriched in $(\infty, n-1)$-categories. We have a kind of structure with objects, and the morphisms between a pair of objects is an $(\infty, n-1)$-category. We could take that as the definition. We want to take categories enriched in $(\infty, 0)$-categories, and these are either topological or simplicial categories. We can again do homotopy theory here, we have a model structure in either case, whether we take simplicial or topological categories. We could think of this as the homotopy theory of $(\infty, 1)$-categories.

There are reasons we might want a different approach. We want to think of things in a weak sense, we don't want things to be strict. You have a strict composition in this case. We want to look at a model where this composition isn't so strict, so we might want a model that's a bit different. So our starting point for the way I'll approach this, is we'll start with a simplicial category, a category enriched in simplicial sets, and we'll take the nerve, which will give us a bisimplicial set.

So this is a functor X from Δ^{op} to simplicial sets. But if it comes from the nerve of a simplicial category, it'll have some special properties. First, X_0 is discrete, we have a discrete set in degree 0 (because we have a set of objects). The second property comes from the composition. The n-simplices are determined by compositions, so if we take the Segal map

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

this is an *isomorphism* of simplicial sets. That's very strong. We're more interested in these things being weak equivalences. So we can consider bisimplicial sets where these maps are just weak equivalences.

This leads to

Definition 1.1. A *Segal category* is a bisimplicial set X such that X_0 is discrete and so that the Segal maps are weak equivalences for every $n \geq 2$.

That gives one way to weaken the composition, but in homotopy theory asking for something to be discrete is awkward. We could, I suppose, ask for it to be homotopy discrete but we'll do something different, leading to the complete Segal spaces of Charles Rezk.

So X_1 is the space of morphisms and X_0 objects. This is enough to make sense of homotopy equivalences, a subspace $X_{heq} \subset X_1$, and we have a degeneracy map $s_0 : X_0 \to X_1$, and the image should be identity maps. This degeneracy factors through this subspace, as it should if these are homotopy equivalences.

Definition 1.2. A (Reedy fibrant) bisimplicial set X is a *complete Segal space* if

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

is a weak equivalence for $n \geq 2$ and

$$X_0 \to X_{heq}$$

is a weak equivalence.

The difference between X_0 and X_1 is the same as the difference between homotopy equivalences and not-equivalences.

I haven't really talked about the homotopy theory of these at all. We'd like model structures but also if this is going to be good we want the homotopy theories to be equivalent.

Theorem 1.1.
1. *(Pelissier, B.) There is a model structure (actually two but I'm shoving this under the rug) I'll call* SeCat *whose fibrant objects are Segal categories (the underlying category is of bisimplicial sets with discrete X_0).*
2. *(Rezk) There is a model structure* CSS *whose fibrant objects are complete Segal spaces (the underlying category is of bisimplicial sets)*

(3) *(B.) There are Quillen equivalences between* (SSets)−Cat*,* SeCat*, and* CSS*.*

How do we continue and generalize from $(\infty,1)$ to (∞,n)? We'd like to find, we have this as a diagram, this is something indexed by Δ, and so maybe we'd like to just replace Δ. Again, we could enrich in SeCat or CSS but then things will be too strict, we'd like to generalize the indexing category instead.

One generalization goes to $(\Delta)^n$, and this gives "multisimplicial" models, and the other way is to use the categories θ_n, I'll call them the θ_n-models, which I'll mainly focus on here. So I first need to tell you what θ_n is.

These categories were first defined by Joyal. I'll follow Berger's θ-construction. Let $\mathcal{C}$ be a small category, and we'll define from it a new category $\theta\mathcal{C}$ which will have objects $[m](c_1,\dots,c_m)$ where $[m]$ is an object of Δ and c_i is an object of $\mathcal{C}$. We draw the object $[m]$, and label the arrows of $[m]$ by c_i. For the morphisms, we have maps from $[m](c_1,\dots,c_m)$ to $[p](d_1,\dots,d_p)$ which are given by maps from $[m]$ to $[m]$ in Δ^{op} along with some other maps $c_i \to d_j$. So for example

$$\begin{tikzcd}
\bullet_0 \arrow[r, "c_1"] \arrow[d] & \bullet_1 \arrow[r, "c_2"] \arrow[d] & \bullet_2 \arrow[r, "c_3"] \arrow[dl] & \bullet_3 \arrow[r, "c_4"] \arrow[d] & \bullet_4 \arrow[dl] \\
\bullet_0 \arrow[r, "d_1"] & \bullet_1 \arrow[r, "d_2"] & \bullet_2 \arrow[r, "d_3"] & \bullet_3 &
\end{tikzcd}$$

and we want maps $c_1 \to d_1$, $c_3 \to d_2$, and $c_3 \to d_3$ in this example.

So we inductively define $\theta_0 = *$ and $\theta_n = \theta\theta_{n-1}$. Note that $\theta_1 = \Delta$. Then θ_2 has objects like $[3]([1],[2],[0])$ where we think of this as having a sequence of 1, 2, or 0 two-morphisms in horizontal sequence

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

We want to consider functors $X\Theta_n^{op} \to \mathrm{SSets}$. Here are two constructions. There's the functor τ_θ from Δ to θ_n which takes $[m]$ to $[m](*,\dots,*)$, where $*$ is the terminal object in θ_{n-1}

This induces a functor $\tau_\theta^* : \mathrm{SSets}^{\theta_n^{\mathrm{op}}} \to \mathrm{SSets}^{\Delta^{\mathrm{op}}}$ which we'll think of as the "underlying bisimplicial set of X."

The second thing that we have in mind is that our (∞,n)-categories should be enriched, maybe in a weak sense, in our $(\infty,n-1)$-categories. So given x,y in $X[0]_0$, we have a mapping object $M_X^\theta(x,y) : \theta_{n-1}^{\mathrm{op}} \to \mathrm{SSets}$ defined by

$$c \mapsto \mathrm{fib}\{X[1](c) \to X[0] \times X[0]\} \text{ over } (x,y)$$

Definition 1.3. A θ_n*-space* is a (Reedy fibrant) functor $X : \theta_n^{\mathrm{op}} \to \mathrm{SSets}$ such that the following hold.

(1)
$$X[m](c_1,\dots,c_m) \to X[1](c_1) \times_{X[0]} \cdots \times_{X[0]} X[1](c_m)$$
for all $m \geq 2$ and c_i in $\mathrm{ob}(\theta_{n-1})$.

(2) The bisimplicial set $\tau_\theta^* X$ is a complete Segal space.

(3) Every $M_X^\theta(x,y)$ is a θ_{n-1}-space for every x and y in $X[0]_0$

Theorem 1.2. *(Rezk) There is a model structure whose fibrant objects are θ_n-spaces.*

We want to know that θ_n-spaces are equivalent to categories enriched in θ_{n-1}-spaces, and we'd like this on the level of model structures.

Theorem 1.3. *(B–Rezk.) There is a chain of Quillen equivalences (we'll mimic the chain for $(\infty,1)$-categories) between $(\theta_{n-1}\text{–Sp})\text{–Cat}$, $\mathrm{SeCat}(\theta_{n-1}\,\mathrm{Sp})$, $\mathrm{CSS}(\theta_{n-1}\,\mathrm{Sp})$, and finally $\theta_n\,\mathrm{Sp}$.*

Let me say some things about these entries. Knowing you have a model structure on categories enriched in another model structure can be difficult in general, but because Rezk's structure is Cartesian and everything is cofibrant, it comes down to checking a condition of Jacob Lurie. I won't say as much about the intermediate stages. So $\mathrm{SeCat}(\theta_{n-1}\,\mathrm{Sp})$ is functors $Y : \Delta^{\mathrm{op}} \to \theta_{n-1}\,\mathrm{Sp}$ satisfying conditions very similar to the Segal category conditions. We want Y_0 to be discrete and $Y_n \to Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1$ should be a weak equivalence in that model structure now.

I want to say more about the complete Segal objects. In analogy with the Segal category objeccts, I can think of these as functors $W : \Delta^{\mathrm{op}} \to \theta_{n-1}\,\mathrm{Sp}$, but I want to think of this as a functor $W : \Delta^{\mathrm{op}} \times \theta_{n-1}^{\mathrm{op}} \to \mathrm{SSets}$. We have a functor, I'll call it τ_Δ this time, which goes $\Delta \to \Delta \times \theta_{n-1}$ which takes $[m]$ to $([m], *)$. This induces $\tau_\Delta^* : \mathrm{SSets}^{\Delta^{op} \times \theta_{n-1}^{\mathrm{op}}} \to \mathrm{SSets}^{\Delta^{op}}$. In fact $\tau_\Delta^*(W) = W(\quad, *)$. Given $x, y \in W([0], *)$, we have $M_W^\Delta(x,y)$ defined as the fiber of $W([1],\quad) \to W([0],\quad)^2$ over (x,y). Using this I want a definition that looks similar to our definition of θ_n-spaces.

Definition 1.4. A (Reedy fibrant) functor $W : \Delta^{\mathrm{op}} \times \theta_{n-1}^{\mathrm{op}} \to \mathrm{SSets}$ is a *complete Segal object* if

(1) (Top level Segal)

$$W([m],c) \cong W([1],c) \times_{W([0],c)} \cdots \times_{W([0],c)} W([1],c)$$

for $m \geq 2$ and $c \in \mathrm{ob}(\theta_{n-1})$

(2) (Top level completeness) $\tau_\Delta^*(W) = W(\quad, *)$ is a complete Segal space

(3) $M_W^\Delta(x,y)$ is a Segal object for every x, y in $W([0],*)$

(4) $W([0],*) \to W([0],c)$ is a weak equivalence for every object c in $\mathrm{ob}(\theta_{n-1})$.

The third item may look a little weird at first because it drops completeness. I'm being a little sloppy, the definition of a Segal object just drops "complete" throughout the definition.

What's the point of the fourth condition? We want just a space of objects, not a more complicated set of objects. We want enriched categories not internal categories. The objects shouldn't have extra structure.

You may ask why this mapping thing shouldn't be a *complete* Segal object? In fact, not just evaluating at a terminal object, evaluating at any object gives a complete Segal space, and that's a *consequence* of the rest of this. So $W(\quad, c)$ is complete.

A last comment, I mentioned the multisimplicial version earlier. We can say something a bit more general about the last Quillen equivalence. We're taking $\Delta \times \theta_{n-1} \to \theta_n$ by $([m], c) \mapsto [m](c, \dots, c)$, and this induces $\theta_n\,\mathrm{Sp} \to \mathrm{CSS}(\theta_{n-1}\,\mathrm{Sp})$. You can keep doing this and get $(\Delta)^n \to \Delta^{n-2} times \theta_2 \to \cdots \to \Delta \times \theta_{n-1} \to \theta_n$ and this gives a way to compare the multisimplicial and θ models.

2. Markus Spitzweck: Hermitian multiplicative infinite loop space machines

Most of this work is joint with Heine and Lopez–Avila, my PhD students. If time permits I'll mention joint work with Niklaus.

The goal is to generate E_∞ ring spectra. We want to mimic what is done for K-theory, so I'll recall a little bit about K-theory. So R is a ring, I denote by $\mathcal{P}(R)$ the category of finitely generated projective R-modules. It's well-known how to get K-theory. So π_0 of this gives commutative monoids with respect to the direct sum operation. This is clearly an associative and commutative operation on the set of isomorphism classes, and the group completion of this is the K_0 group of R, the group completion of $\pi_0\mathcal{P}(R)$. Because we're interested in ring structures, I should know that R is commutative. We then also have the tensor product over R of projective finitely generated R modules. This endows $\mathcal{P}(R)$ with a second symmetric monoidal structure and we get a commutative ring structure on $K_0(R)$.

What about higher K groups? Well $\mathcal{P}(R)$ together with $\oplus$ and the 0 module is equivalent to the so-called permutative category. Then I take the nerve of the groupoid of $\mathcal{P}(R)$, and that's an E_∞ space. You can let the Barratt–Eccles operad act on it. How do I take the higher K-groups? I take the group completion, this is a group-like space. This is the K-theory space of R. This is a grouplike E_∞ space so it corresponds to a connective spectrum $K(R)$. Then the higher K-groups are the homotopy groups of this spectrum.

If R is again commutative, then $\mathcal{P}(R)$ can be strictified to a *bipermutative* category. There is machinery by May–Ellmendorf–Mandell to get an E_∞ ring spectrum with underlying spectrum [unintelligible]. Then $K(R)$ is an E_∞-spectrum, then the goal is to produce something like this in a Hermitian context.

If $\mathcal{C}$ is an ∞ category with finite products, then I denote by $\mathrm{Mon}_{E_\infty}(\mathcal{C})$ the ∞-category of commutative monoids in $\mathcal{C}$, and you can take the functor category from pointed finite sets to $\mathcal{C}$ with the properties, the full subcategory so that $F(\langle n\rangle = * \sqcup \{1,\dots,n\}) \cong F(\langle 1\rangle)^n$, these maps should be equivalences.

Now notation Cat_∞ is the ∞-category of small ∞-categories. The category of $\mathrm{Sym\,Mon\,Cat}_\infty$ is the E_∞-monoids in Cat_∞.

You can also look at $\mathrm{Grp}_{E_\infty}(\mathcal{C})$ in $\mathrm{Mon}_{E_\infty}(\mathcal{C})$, there are several ways to say this, and if you specialize to spaces, then $\mathrm{Mon}_{E_\infty}(\mathrm{Spc}) \cong E_\infty$-spaces and $\mathrm{Grp}_{E_\infty}(\mathrm{Spc})$ is the subcategory of grouplike E_∞ spaces.

So let me talk aobut direct sum K-theory. I have

$$K : \mathrm{Sym\,Mon\,Cat}\,\infty \xrightarrow{(\ )^\sim} \mathrm{Mon}_{E_\infty}(\mathrm{Spc}) \to \mathrm{Grp}_{E_\infty}(\mathrm{Spc}) \to \mathrm{Sp}$$

Work of Gepner–Groth–Nikolaus says that $\mathrm{Sym\,Mon\,Cat}_\infty$ itself has a closed symmetric monoidal structure and K is lax symmetric monoidal. This has the advantage that it preserves all kinds of algebra objects, in particular E_∞-algebras.

Then $\mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)$ is E_∞ objects in $\mathrm{Sym\,Mon\,Cat}_\infty$, which is $\mathrm{Alg}_{E_\infty}(\mathrm{Sym\,Mon\,Cat}_\infty)$. Here this is a categorification of rigs, which have no additive inverses but are otherwise rings.

How do we produce these?

Theorem 2.1. *(Gepner–Groth–Nikolaus) Let $\mathcal{C}$ be a symmetric monoidal ∞-category with finite coproducts and so that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves coproducts in each variable. Then one can turn the coproduct into a symmetric monoidal structure, and so $\mathcal{C} \in \mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)$.*

In a little while I'll discuss the proof because we want to extend to the Hermitian case.

As an example, the category $\mathcal{P}(R)$ is in $\mathrm{Rig}_{E_\infty}(\mathrm{Cat})$ (since it's an ordinary category). For R in $E_\infty(\mathrm{Sp})$ we have finitely projective R-modules $\mathcal{P}(R)$, and this is in $\mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)$. It follows that you get a K-theory E_∞ ring spectrum. You have to be careful. This category $\mathcal{P}(R)$ only sees the connective part of R. If R is connective then $K(\mathcal{P}(R))$ is the usual K-theory of R considered by many people. Otherwise you get [unintelligible].

Now we come to Hermitian K-theory. A small introduction. I consider a commutative ring for simplicity. Then $\mathcal{P}(R)$ has a duality $M \to M^\vee$. Then a Hermitian object is $M \xrightarrow{\varphi} M^\vee$, an isomorphism, and you have $M \to M^{\vee\vee}$, and then you can take the dual $M \to M^{\vee\vee} \to M^\vee$ and this should also be φ. This information is the same as a symmetric nondegenerate bilinear form on M. Then you have an orthogonal sum $M \oplus N \rightsquigarrow \mathcal{P}(R)_h$, taking the direct sum of the underlying objects. Then we have a symmetric monoidal structure and can group complete an E_∞ space.

But you can't apply the theorem, this is neither the coproduct or the product. But anyhow we can take $\mathcal{P}(R)_h^\sim$, the group completion, which again correpsonds to a connective spectrum, $K_h(R) \in \mathrm{Sp}$. This goes back to Karoubi (this is in the case when 2 is invertible). There is recent work saying that we do not actually need 2 to be invertible.

Because R is commutative you also have the tensor product of objects. We want to mimic the approach of Gepner–Groth–Nikolaus. You can't apply their theorem directly because it's not the coproduct.

Let me recall the following theorem, in this version due to Toën (but there are also higher versions due to Barwick–Schommer-Pries)

Theorem 2.2. *The A_∞ space of autoequivalences of Cat_∞ is $\mathbb{Z}/2$ and the nontrivial element, as one expects, acts as $C \mapsto C^{\mathrm{op}}$.*

This is not possible if Cat_∞ were being considered with 2-morphisms, but this works here.

Now Cat_∞ is in $\mathrm{Pr}^L[C_2]$, the presentable ∞-categories with a C_2-action, and you can take the so-called homotopy fixed points, this can be seen as a functor $\mathrm{Cat}_\infty^{hC_2}$, it's the ∞-category of small ∞-category with duality.

This isn't the nicest definition. A C_2 space should have the fixed point information as datum.

We have one small proposition, which says that duality on C corresponds to C_2-homotopy fixed points of $Fun(C \times C, \mathrm{Spc})$, and you want the underlying functor to be a perfect pairing. That's an observation. We have some other observations. Inside Cat_∞, the subcategory of Spc, the C_2 action is trivial.

Further, $\mathrm{Sym\,Mon\,Cat}_\infty$ and $\mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)$, these all inherit C_2 actions. The C_2 homotopy fixed points preserve the monoidal structures. Then $E_\infty(\mathrm{Sym\,Mon\,Cat}_\infty^{hC_2})$ is equivalent to $\mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)^{hC_2}$, you have several such equations.

Now we can define the direct sum version of Hermitian K-theory, this is the following functor, we can start with an additional C_2-action,

$$K_h : \mathrm{Sym\,Mon\,Cat}_\infty^{hC_2} \xrightarrow{(\ )^\sim} \mathrm{Mon}_{E_\infty}(\mathrm{Spc})[C_2] \xrightarrow{(\ )^{hC2}} \mathrm{Mon}_{E_\infty}(\mathrm{Spc}) \to \mathrm{Grp}_{E_\infty}(\mathrm{Spc}) \to \mathrm{Sp}$$

Proposition 2.1. *K_h is lax symmetric monoidal.*

So we want to look at the subcategory of ∞-categories with coproducts (non-full, with coproduct preserving functors), this $\mathrm{Cat}_\infty^\Sigma$ is *preadditive*, with a null object, products and coproducts, and now we consider the following functor

$$\mathrm{Sym\,Mon\,Cat}_\infty \cong \mathrm{Mon}_{E_\infty}(\mathrm{Cat}_\infty) \to \mathrm{Mon}(\mathrm{Cat}_\infty^\Sigma) \cong \mathrm{Cat}_\infty^\Sigma$$

and all these functors are symmetric monoidal. There is a right adjoint to this composition, one can equip $\mathrm{Cat}_\infty^\Sigma$ with a monoidal structure, I should have said. [missed a little]. Then applying the right adjoint we get the required structure.

Now we want to include everything with dualities. If you take the opposite of an ∞-category with coproducts it has products but not coproducts. So we consider $\mathrm{Cat}_\infty^{\mathrm{preadd}}$, which sits inside $\mathrm{Cat}_\infty^\Sigma \to \mathrm{Sym\,Mon\,Cat}_\infty$, this is lax symmetric monoidal, with C_2-actions. This then gives a natural functor

$$(Cat_\infty^{\mathrm{preadd}})^{hC_2} \to \mathrm{Sym\,Mon\,Cat}^{hC_2}$$

Now we come to the second question, how do we get the duality? That's not a trivial question, and there I should state a theorem

Theorem 2.3. *(H.–L.-A.–S.) Look at the* rigid *symmetric monoidal categories, the homotopy fixed points, then the forgetful functor, this has a natural section. A rigid symmetric monoidal category can be equipped with a natural duality respecting the symmetric monoidal structure, this is the formual* $(X \otimes Y)^\vee = Y^\vee \otimes X^\vee$

Now we are in the position to put things together.

We will get by combining this theorem with previous statements the following. If we have a symmetric monoidal rigid category which is preadditive in the sense that the sum distributes over the symmetric monoidal structure, then we get an object in $\mathrm{Rig}_{E_\infty}(\mathrm{Cat}_\infty)^{hC_2}$. The monoidal structure is the second one and the preadditive is the first one. Then we can apply $K_h(C)$ and get an E_∞ ring spectrum.

For example, we can put in R an E_∞-spectrum, then we have $\mathcal{P}(R)$ which has a rigid preadditive structure, so we get, $K_h(R)$ is an E_∞ ring spectrum.

In the end I would like to comment a little bit on ongoing work.

There are different dualities, you can twist this by tensoring with [unintelligible]objects. There is a functor $\mathrm{Pic}(C) \to \mathrm{Cat}_\infty^{hC_2}$, lax symmetric monoidal, and the duality is given by taking a line bundle L, $X \mapsto X^\vee \otimes L$. Because this is lax symmetric monoidal, this is a module over the original E_∞ ring spectrum.

You can equip everything with additional group actions, so for example start with $\mathrm{Pic}(C)^{hC_2}$, and maybe you get $\mathrm{Pic}(C)[C_2]$, so $\mathbf{1}[1]^{\otimes 2}$, you get symplectic Hermitian K-theory. That's one thing. You also have another theorem which states that for an arbitrary line bundle L, take a twisting c on $L^{\otimes 2}$. alread the duality will be the same as $\mathbf{1}, \mathrm{id}$, this is a periodicity.

I wanted to draw a diagram but I'll just mention. In the case of preadditive Cat_∞ with duality, there is a [unintelligible]spectrum with fixed points this [unintelligible]and underlying [unintelligible]the K-theory of [unintelligible]. This is a version of real K-theory in this context, ongoing with Thomas Nikolaus. It's even possible to equip this with an E_∞ structure.

3. March 19: Ben Ward: Feynman categories

It's a pleasure to be here in Bonn for the first time. This is joint work with Ralph Kaufmann, and the goal is to talk about operads, or work with them, and their generalizations, using the language of category theory. I think it's important for me to say that this is one approach to this problem, so what are the advantages of this approach? I can summarize the advantages from my perspective: "it's just category theory." The basic tools are familiar to many mathematicians. Let me be a bit more specific.

(1) We recover familiar constructions and extend them using basic category theory.
(2) We get some new constructions that might not arise from other formalisms.
(3) Pedagogical advantage—if someone asks what an operad is, they don't know, then you don't want to tell them that an operad is an algebra over a certain colored operad.

Let me give encoding structure. I'll define a category $\mathbb{V}$ whose objects are $\mathbb{N}$ and with only automorphisms, $\mathrm{Aut}(n) = S_n$. Then I'll define another category $\mathbb{F}$, a symmetric monoidal category. The objects of $\mathbb{F}$ are lists from $\mathbb{V}$, tensor generated by $\mathrm{ob}\,\mathbb{V}$. The morphisms are $\otimes$-generated by graphs formed out of rooted trees assembled from the source.

So for example [pictures]. Morphisms compose by grafting trees.

Definition 3.1. An operad in a symmetric monoidal category $\mathcal{C}$ is a symmetric monoidal functor $\mathbb{F} \to \mathcal{C}$.

That is you get for each n an object $\mathcal{O}(n)$ with a strict S_n action and then operations $\mathcal{O}(T) \to \mathcal{O}(n)$.

This is an example of a Feynman category. Now I'll give a definition.

Definition 3.2. A Feynman category is a 3-tuple $(\mathbb{V}, i, \mathbb{F})$, where $\mathbb{V}$ is a groupoid, i is a functor from $\mathbb{V}$ to $\mathbb{F}$, and $\mathbb{F}$ is a symmetric monoidal category. This data should satisfy the following requirements:

(1) The functor $i^{\otimes} : \mathbb{V}^{\otimes} \to \mathrm{Iso}(\mathbb{F})$ is an equivalence of symmetric monoidal categories
(2) $\mathrm{Iso}(\mathcal{F} \downarrow \mathbb{V})^{\otimes} \xrightarrow{i^{\otimes}} \mathrm{Iso}(\mathbb{F} \downarrow \mathbb{F})$ is an equivalence of symmetric monoidal categories.

To specify an $\mathbb{F}$, I need a set of basic objects, which are the objects of $\mathbb{V}$, call them v_i or "vertices." Then I need objects of $\mathbb{F}$, which are given up to isomorphism by lists of objects of $\mathbb{V}$. For my purposes today I'll say it's a list of vertices. I need to specify morphisms of $\mathbb{F}$. A basic morphism looks like $v_1 \otimes v_n \to v_0$, this is like a graph assembled from the list of vertices. There's a close relationship to colored operads, where v_i represent the colors.

Definition 3.3. The category $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$ (here $\mathcal{C}$ will be a symmetric monoidal category with whatever properties I want) is monoidal functors from $\mathbb{F}$ to $\mathcal{C}$. I'll call these "$\mathbb{F}$-ops." I can also let $\mathbb{V}\text{-}\ \mathrm{mod}_{\ \mathcal{C}}$ by functors from $\mathbb{V}$ to $\mathcal{C}$, or "$\mathbb{V}$-mods."

I've got some examples here [pictures] for an $\mathbb{F}$ such that $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$ is the category of operads, non-Σ operads, (non-Σ) cyclic operads, modular operads, genus zero dioperads, props or properads depending on connectedness, $\frac{1}{2}$-props, wheeled prop(erads), and now you get back to point one, you define this category and take functors if you don't know what an operad is. We all maybe know operads but this works for more esoteric structures.

Let me discuss two other very simple examples. For any groupoid V, I can just take $\mathbb{V}^{\otimes}$ as $\mathbb{F}$. I can also, if $\mathbb{V}$ has just one object, then the data of an operad in sets is the same thing (maybe up to equivalence) as triples with that $\mathbb{V}$. Then algebras over these operads are the same thing as $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$.

"The things we want to do are just category theory." Let me say some of what I mean.

(1) You want free or enveloping constructions. It turns out that all of these guys can be obtained by Kan extensions.
(2) Another thing you'd like to do is to have model structures on the categories of $\mathbb{F}-$ops, and it turns out that a key argument here is that the requirements for transfer are symmetric monoidal.
(3) We want natural operations, something along the lines of graph complexes. Because these operads are functors I can take limits and colimits.
(4) Bar constructions or Feynman transforms, essentially equivalent to master equations.
(5) W-construction (both of these are about getting cofibrant replacements)
(6) Because I have these Kan extensions, I get duality, dual extensions. That's a lot of structure, and I can exploit this by understanding intertwining or preservation of Koszulity for these guys.

These first three things make no requirements on the initial Feynman category. Starting with the fourth I need a notion of "edges."

So first of all, free constructions or envelopes. The setting is we have a morphism between two Feynman categories, exactly what you think it should be. Remember that $\mathcal{C}$ is nice, it's got colimits which commute with tensor in each variable, say Cartesian closed. So given $\mathbb{F}_1 \xrightarrow{\pi} \mathbb{F}_2$ we get an adjunction $\mathbb{F}_1-\mathrm{ops}_{\mathcal{C}}$ and $\mathbb{F}_2-\mathrm{ops}_{\mathcal{C}}$ and we have a left adjoint from the former to the latter, L. I can take a left Kan extension, it's a colimit over maps from $\pi \downarrow v$, $\mathcal{O} \circ s$, this is a pointwise Kan extension. Some examples, for example if I go between $\mathbb{V}-\mod$ and $\mathbb{S}-\mathrm{ops}$, this is the free operad on an $\mathbb{S}$-module.

Let me give an example with a nice property. Take $\mathbb{F}_1$ to be dioperads, directed genus zero, and take the obvious morphism to $\mathbb{F}_2$, which is cyclic operads, I just forget directions to get π. This gives us an adjunction, with left adjoint from dioperads to cyclic operads. In this case, the category $\pi \downarrow v$ is a disjoint union of filtered categories. I'm taking a colimit out of a category like this, then in an Abelian category L is levelwise exact. That's something, basically using a classical fact from category theory, we get a little bit more.

While I'm erasing the board let me say a little bit about why $\frac{1}{2}$-props. This is due to Markl and Voronov. They make a statement like this about $\frac{1}{2}$-props and props.

Let me talk about model structures now.

Theorem 3.1. *For nice $\mathcal{C}$ (differential graded vector spaces in characteristic zero, simplicial sets, topological spaces), $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$ is a model category where R creates fibrations and weak equivalences.*

I don't want to prove this but let me tell you what this involves (standard techniques about transfer across adjunctions). These are between $\mathbb{V}-\mod_{\mathcal{C}}$ and $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$ and between $\mathbb{V}-\mod_{\mathcal{C}}$ and $\mathbb{V}-$sequences, where all the conditions for transfer can be pushed down to $\mathcal{C}$. For example, I need a path object functor for fibrant objects in $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$, but I will have this if I have a symmetric monoidal path object in $\mathcal{C}$. Berger–Moerdijk posit a cocommutative interval object. From this perspective it makes sense, since $\mathrm{Hom}(I, \quad)$ is your path object.

You can use these to get Quillen adjunctions, and bar constructions or Feynman transforms are cofibrant objects.

Now let me talk about natural operations. Maybe I'll just do the general story. The general story, starting with $\mathcal{O}$ in $\mathbb{F}-\mathrm{ops}_{\mathcal{C}}$, I get two natural objects associated to $\mathcal{O}$, namely the limit of $\mathcal{O}$ and the colimit of $\mathcal{O}$. These will be algebras over some operad, and the question is whether you can construct that operad. So $\mathrm{colim}(\mathcal{O})$ is an algebra, I can take the limit over length n of the colimit over $\mathbb{V}$ of the morphism sets $\mathrm{Hom}(i(\quad)^{\otimes n}, i(\quad))$. This thing could be huge or it could be trivial, and we'll see it serves as a receptacle for operations that we know, expect, and want to generalize, like brace operations for the differential in the bar construction which in this guise play the role of the Lie bracket. It's interesting to compute this after twisting. If you do this for operads you get Lie, for non-symmetric operads a solution for the Deligne conjecture, for cyclic operads you get a model for gravity.

This is what you can do for an arbitrary Feynman category. When the morphisms have edges that can be contracted in every possible way, you can formulate it in an even and odd way, you can make a W construction, and the intertwining is still in progress. I'm out of time so let me stop there.

4. March 20: Alexander Voronov: Categorification of Dijkgraaf–Witten theory

This is joint with Amit Sharma. I tood this picture of nameplates saying K. J. Korteweg and A. E. M. de Vries. It was actually an office of lawyers. Somehow those two guys come in this order.

Martin Markl told me that in every talk there should be something entertaining, and this is the thing in my talk, even though it's not related.

So I'll be talking about a categorification of Dijkgraaf–Witten theory. That might not be the right way to call it. It's some kind of categorification, but what Ralph Kaufmann did with [unintelligible]in 2009 is more like a real categorification. So anyway, this is a TQFT coming from a gauge theory with finite gauge group G. These days, even though it was created in 1990 in a paper of Dijkgraaf and Witten, there's renewed interest these days with extended topological quantum field theories, even this kind of toy model. In relation to ambidexterity as well there's renewed interest.

In a sense this reminds me of a phrase I saw in the streets of Moscow in the Soviet Union: "Marxist doctrine is omnipotent because it's true." This was a quote of Lenin. So to me it feels like Dijkgraaf–Witten theory is omnipotent because it's true.

So you start with a finite group G and a free cohomology class $\alpha \in H^3(BG, U(1))$, where $U(1) = \mathbb{R}/\mathbb{Z}$ (or I could take $\mathbb{Q}/\mathbb{Z}$). This is isomorphic to $H^4(BG, \mathbb{Z})$, but you better work with a 3-cocycle. You create a 3-dimensional TQFT out of this by assigning to a two dimensional manifold Y the vector space $\phi(Y)$. To a 3 dimensional (oriented) cobordism, you assign $\phi(X)$ from $\phi(\partial_-(X))$ to $\phi(\partial_+(X))$, a linear map. The way they constructed it is mysterious, even though the formulas are well-understood. The basic ingredients, though, were not quite well-understood.

The dimension of $\phi(Y)$ is less than or equal to $|\operatorname{Hom}(\pi_1(Y),G)/G|$, this is finite, quotiented by conjugation of G, this is isomorphism classes of principal G-bundles over Y. It's not quite equal. You might start wondering what the space is.

They do the typical physical trick. They assume that $\phi(Y)$ is constructed and then compute the dimension of $\phi(Y)$ by taking the trace of the operator $\varphi(Y \times S^1)$. You get an operator from $\mathbb{C}$ to $\mathbb{C}$. [picture]

You want a map for each cobordism so you might want something more than the dimension.

This they provide a formula for this, if X is a cobordism, then $\phi(X)(\gamma)$ is

$$\frac{1}{|G|}\sum_{\gamma'':\gamma''|_{\partial_-(X)}=\gamma} W(\gamma'')\gamma''|_{\partial_+(X)}$$

where we consider γ'' as something like a principal G bundle over X, or $\operatorname{Hom}(\pi_1(X),G)$. This assumes that the space is related to the characteristic variety of X or of $\partial_\pm X$, but not quite.

Here $W(\gamma'')$ is defined as $\int_X (\gamma'')^*(\alpha)$, and now we think of γ'' as a map $X \to BG$, if you have a principal G-bundle you have such a map defined up to homotopy. You pull back and then take the integral.

There are some limitations from this. It's not clear how to identify $\phi(Y)$ as a vector space. Also $\operatorname{Hom}(\pi_1(Y),G)/G$ is identified with principal G-bundles over Y, and this is in turn identified with $[Y,BG]$. We want to treat this as an orbifold rather than just a set. The orbifold is *finite*, a finite set quotiented by a group on the left, but then the principal G-bundles is infinite dimensional.

In 1993, Freed–Quinn answered all of these questions and constructed this field theory, the space $\phi(Y)$ and $\phi(X)$ as morphisms of vector spaces. They also generalized it to $n+1$-dimensional field theories, and they used the following interesting pairings between cocycles valued in $U(1)$ on the n-dimensional manifold Y and the $n+1$-dimensional cycles of Y with coefficients in $\mathbb{Z}$ to $U(1)$. The same is true in X the cobordism. This can't be done naively because there are no $n+1$-dimensional classes.

Let me talk about how Lurie treated this in 2012. He worked with $\operatorname{Map}(Y,BG)$ and considered $\alpha \in H^{n+1}(BG,U(1))$, resolved the problem of integration. He said, let's define, take the evaluation map $Y^n \times \operatorname{Map}(Y,BG) \to BG$. You want an $n+1$-dimensional field theory. He took the projection on the second factor and then direct image of the inverse image of α, $\pi_* \operatorname{ev}^* \alpha$ (integrating out the fiber Y) and you get an element in $H^1(\operatorname{Map}(Y,BG),U(1))$, and this is a flat Hermitian line bundle over the mapping space $\operatorname{Map}(Y,BG)$, and for $\phi(Y)$ you take H^0 with coefficients in this line bundle (a linear system on this space) $H^0(\operatorname{Map}(Y,BG),\mathcal{L}_Y)$.

For $(n+1)$-dimensional objects, Lurie used ambidexterity to identify $H^0(\operatorname{Map}(Y;BG),\mathcal{L}_Y)$ with $H_0(\operatorname{Map}(Y,BG),\mathcal{L}_Y)$ and then given a cobordism you construct $\phi(X):\phi(\partial_-(X)) \to \phi(\partial_+(X))$; of course you have the span $\operatorname{Map}(\partial_- X,BG) \leftarrow \operatorname{Map}(X,BG) \to \operatorname{Map}(\partial_+ X,BG)$, these are restrictions of the map from the cobordism to its boundaries.

You define $\varphi(X)$ as the pullback in cohomology of p_- and then the pushforward in homology along p_+. This map is in some sense $p_-^* A(p_+)_*$ where A is the ambitexterity map.

There's a subtlety, $\mathcal{L}_Y$ is determined by $\pi_* \operatorname{ev}^* \alpha$ only up to isomorphism, but you need a natural vector space, this depends on the choice of the line bundle. You want to have a genuine line bundle out of this construction but the construction doesn't give it to you.

In Lurie's paper, he remarks that in order to cope with this problem, he uses a cocycle instead of a cohomology class for α. Then you obtain an honest to goodness cocycle, this isn't the *same* as a line bundle (and again only determines it up to isomorphism), and a line bundle just isn't the same as a cocycle.

So we felt like there was a need for a more categorical construction and this is what we've done. Also another reason why we wanted to categorify this construction is to supply the extension to a field theory out of [unintelligible]data. The goal is to construct an extended field theory that starts with a finite group and some cocycle data on it. Before that we wanted to just do the Dijkgraaf–Witten theory the right way. We wanted honest to goodness line bundles and vector spaces corresponding to n-dimensional things, et cetera.

This is what I'll describe in the remaining half of my talk.

So here's the idea of our construction, which we posted a year ago on the arXiv. The initial data is α in $H^n(BG, \mathcal{L})$, with coefficients in the Picard groupoid of Hermitian lines (I'll tell you in a minute what this is) rather than $H^{n+1}(BG, U(1))$.

What is this $\mathcal{L}$? It's a groupoid, a Picard groupoid is a categorification of an Abelian group. It's a symmetric monoidal category in which every object is invertible up to isomorphism with respect to the tensor product, which I will denote $+$. The Picard groupoid of Hermitian lines has as objects Hermitian lines, one dimensional complex vector spaces with a Hermitian form. These are the objects.

Morphisms between two lines are linear isometries $\operatorname{Iso}(L_1, L_2)$, a $U(1)$ worth of morphisms for each pair, but a $U(1)$-torsor because you don't have a distinguished isomorphism between them.

If $\mathcal{A}$ is a Picard groupoid, you can associate to it $\pi_0(\mathcal{A})$, the Abelian group of connected components, and $\pi_1(\mathcal{A})$, the group of automorphisms of the 0 object, which turns out to be an Abelian group because of the Eckmann–Hilton principle.

Further examples of Picard groupoids, you can associate two Picard groupoids to any Abelian group, the discrete one $A[0]$ and the one with one point $A[1]$. For $A[0]$ the objects are elements of A and the morphisms are the identity of a if $a = b$ and otherwise nothing. Another has a single object and the morphisms between that object and itself is A.

A second example is $\mathcal{L}$, the Picard groupoid of Hermitian lines.

Now I want to use cohomology with coefficients in the Picard groupoid. If you have a simplicial set X or topological space, and Picard groupoid $\mathcal{A}$, then you associate Picard groupoids $H^n(X, \mathcal{A})$. This theory was actually created by the Spanish school, Carrasco and Martinez [unintelligible]in 2000. An n-cocycle is a pair (c, α) where $dc \xrightarrow{\alpha} 0$, where the thing is, if you define a complex $C^*(X., \mathcal{A})$, it's a 2-complex $0 \to \mathcal{A}^{X_0} \to \mathcal{A}^{X_1} \to \mathcal{A}^{X_2} \to \cdots$ and there are maps d, connecting morphisms, but the morphism don't square to zero, they only square to zero in a 2-categorical sense, there is a 2-morphism from d^2 to 0.

I should say that $\pi_0 H^n(X; \mathcal{A})$ is isomorphic to $H^{n+1}(X, \pi_1(\mathcal{A}))$. For example if $\mathcal{A}$ is $\mathcal{L}$ then $\pi_0 H^n(X, \mathbb{L}) \cong H^{n+1}(X, U(1))$ so this is some kind of delooping at the level of coefficients. This allows us to make the cohomology group smaller, the pairing of Z^{n+1} and C_{n+1} becomes a pairing of [unintelligible]with the fundamental class of Y.

We defined a cap product $H^n(X,\mathcal{A})\otimes H_k(X,\mathbb{Z}[0])\to H_{k-n}(X;\mathcal{A})$.

We define a Dijkgraaf–Witten theory. For $f:Y\to BG$, for a continuous map $Y\to BG$ we can pull back α along f, and $f^*(\alpha)$, you cap it to $H_n(Y;\mathbb{Z}[0])$ and the result will be in $H_0(Y,\mathcal{L})$, and inside $H_n(Y;\mathbb{Z}[0])$ is a subgroupoid C_Y of representatives of the fundamental class $[Y]$. You then have a functor $C_Y\to H_0(Y,\mathcal{L})$, you take the limit of this functor, you get a line bundle that you associate, just a second, you get a line bundle, just a line so far, but it turns out to make up a line bundle over the mapping space from Y to BG. Since we did it for each map F, we get a functor $\mathcal{L}_Y:\pi_1(\mathrm{Map}(Y,BG))\to\mathcal{L}$ and then $\phi(Y)=H^0(\mathrm{Map}(Y,BG),\mathcal{L}_Y)$. This gives a vector space which is $\phi(Y)$ and you also get $\phi(X)$, where X is a cobordism, the idea is similar to Lurie's suggestion.

The theorem is that this indeed gives a topological quantum field theory as suggested.

5. March 21: Boris Tsygan: A microlocal category associated to a symplectic manifold

So thank you very much for the invitation, I'm delighted to come here and give a talk. So I will describe some construction, how to construct some sort of a category. I will specify it, for a symplectic manifold, maybe with a topological condition. Let me start by saying if you have (X,ω) then the category will be some sort of "improved" modules over the deformation quantization algebra $C^\infty(X)[[\hbar]]$. Let me tsart by trying to explain this. What is the deformation quantization? Let me be very brief here and say it's definitely a sheaf of algebras over $\mathbb{C}[[\hbar]]$ on X and it will have the following property, locally in some Darboux coordinates (and maybe one should take a little care with which coordinates),

$$f *_W g=\sum_{n\geq 0}\frac{(i\hbar)^n}{n!}(\partial_\xi\partial_y-\partial_\eta\partial_x)^n f(x,\xi)g(y,\eta)|_{x=y,\xi=\eta}$$

Very important is that this is associative and $f*g-g*f=i\hbar\{f,g\}+\hbar^2\cdots$ (and up to $\hbar$ it's the ordinary commutative product). It's important that $*_W$ is $Sp(2n)$-equivariant. If you change coordinates linearly, it's important that you get the same algebra.

Naively one's first guess (I heard it from Boris [unintelligible]thirty plus years ago) is that one should consider the category of sheaves of modules over this sheaf of algebras, and that has intriguing formal similarities with the Fukaya category. One sees quickly (as Boris also did) that this is way too naive. So maybe there's an improvement to make this, not the Fukaya category, but let's say closer to the Fukaya category.

So we want to improve this along the lines of allowing not only $f(x,\xi,\hbar)$ but also $e^{\frac{f(x,\xi)}{i\hbar}}$.

The first step is, it already has this flavor as Fedosov realized, but sort of artfully avoided. Instead of my sheaf of algebras (let me call it $\mathbb{A}$ or $\mathbb{A}_X$), let me replace it by $\hat{\mathbb{A}}=C^\infty(U)[[\hat{x},\hat{\xi},\hbar]]$ for U open in X, and consider smooth expressions with values in these formal functions. Equip it with the Weyl product with respect ot $\hat{x}$ and $\hat{\xi}$ and $\hbar$. In x and ξ it's just the ordinary product. This is a much much bigger thing, but there is a flat connection on it, let me write $\hat{\mathbb{A}}^\bullet = \Omega^\bullet \otimes_{C^\infty} \hat{\mathbb{A}}$. So locally

$$\hat{\mathbb{A}}^\bullet(U) = C^\infty(x,\xi)[[\hat{x},\hat{\xi},\hbar]]\langle dx, d\xi\rangle$$

and it comes with this flat connection

$$\nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right)dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right)d\xi$$

and the beauty of this flat connection is it cuts it down to size, the zero homology is, maybe I should say, well $f(x,\xi)$ embeds into this by $f(x+\hat{x},\xi+\hat{\xi})$, and it's obviously flat, this is a quasiisomorphism $\mathbb{A} \hookrightarrow \hat{\mathbb{A}}^\bullet, \nabla_{\mathbb{A}}$.

Fedosov showed that such a thing always exists, and so the second naive guess is sheaves of differential graded modules $(\mathbb{V}^\bullet, \nabla_V)$ over $(\hat{\mathbb{A}}^\bullet, \nabla_{\mathbb{A}})$. But still, include $e^{\frac{f}{i\hbar}}$? We want to include these. What I'h like to do is replace $\mathbb{C}[\hat{x},\hat{\xi},\hbar]$ by something bigger. Let me forget x and ξ for a second and just look at the formal variables at an individual point and try to realize what we want to achieve.

So $\mathbb{C}[\hat{x},\hat{\xi},\hbar]$, we want to expand this to A which includes $e^{\frac{f}{i\hbar}}$ Let me notice that there is a grading here where $|\hat{x}| = |\hat{\xi}| = 1$ and $|\hbar| = 2$, this makes sense since $[\hat{\xi}_j, \hat{x}_k] = \delta_{jk} i\hbar$. Let me introduce denominators. Fedosov artfully avoids all denominators, but I have to introduce them. So I consider

$$\mathbb{C}\{\hat{x},\hat{\xi},\hbar\} = \{\sum_{\substack{k\in\mathbb{Z}\\ m,n\geq 0\\ m+n+2k\to\infty}} a_{mnk}\hat{x}^m\hat{\xi}^n\hbar^k\}$$

So monomials $\hat{x}^m\hat{\xi}^n\hat{\hbar}^k$, wehre $m,n\geq 0$ and $k\in\mathbb{Z}$. The degree of this monomial is $m+n+2k$. I'll consider countable sums $\sum_p a_{\mathbb{X}_p}\mathbb{X}_p$ where the degree goes to ∞.

By this innocent completion, it's still a ring, $\mathbb{C}\{\hat{x},\hat{\xi},\hbar\}$ still has an action of $Sp(2n)$, and it contains $e^{\frac{f(\hat{x},\hat{\xi})}{i\hbar}}$ if

$$f = \sum_{n\geq 3} f_n(\hat{x},\hat{xi}).$$

So what remains? Costant, linear, and quadratic. So

$$\mathbb{K} = \sum_{\substack{n=0\\ c_n\to\infty}}^{\infty} a_n e^{\frac{c_n}{i\hbar}}$$

this lets you add constant terms, you just add these by forcing. What about linear and quadratic? Let's start with quadratic. We should not take this one hundred percent literally. What I would do is, I'd take $\mathbb{K}[\widetilde{Sp}(2n)]$ as a *discrete* group. We take the semidirect product

$$\mathcal{A} \coloneqq \mathbb{K}[\widetilde{Sp}(2n)]\hat{\ltimes}\mathbb{C}\{\hat{x},\hat{\xi},\hbar\}$$

and we need some mild condition with respect to the product, so just sort of completed a little with respect to the same kind of thing.

The motivation is if we consider

$$\mathfrak{sp}(2n) \cong \{q(\hat{x},\hat{\xi})/i\hbar | q \text{ quadratic}\}$$

and this is the infinitesimal action. So these things in $\widetilde{Sp}(2n)$ correspond to the quadratic terms. So this is my algebra $\mathcal{A}$.

Now $\mathcal{A}^\bullet_X$ is $\mathcal{A}$-valued forms on X. Locally there will be expressions

$$\sum a_{\dots}(x,\xi)e^{\frac{\varphi_{\dots}(x,\xi)}{i\hbar}}[g_{\dots}(x,\xi)]\hat{x}^m\hat{\xi}^n\hbar^k dx_I d\xi_J$$

where I'll say maybe $\cdots = mnk$ and several φ may correspond to one monomial.

Maybe what's important to stress is that,

- it's a perfectly well-defined sheaf of graded algebras,
- the differential extends naturally to a differential $\nabla_{\mathbb{A}}$ here, and
- this is not quasicoherent sheaves over C^∞_X, [unintelligible]the good notion of f^* for $f : X \to Y$.

Then a strange and important thing appears.

Again, there are maybe some minor topological conditions, it works better if c_1 is zero. But the fact is that $\pi_1(X)$ acts on $\mathcal{A}^\bullet$ up to inner automorphisms.

First of all, what does it mean? Let's say you have a groupoid $\mathfrak{g}$ with objects X, and $\mathcal{A}$ a sheaf of algebras on X. Then we say $\mathfrak{g}$ acts on $\mathcal{A}$ up to inner automorphisms if there is an action $\mathfrak{g} \times t^*\mathcal{A} \to s^*\mathcal{A}$ where for local sections $g, a \mapsto T_g a$ by algebra isomorphisms for each g, but then $T_{g_1} T_{g_2} = Ad_{c(g_1 g_2)} T_{g_1, g_2}$, to make it multiplicative in g you eed an inner automorphism $c(g_1, g_2) \in s^*\mathcal{A}^\times$.

But then $c(g_1, g_2)c(g_1 g_2, g_3) = T_{g_1} c(g_2 g_3) c(g_1, g_2 g_3)$ in $\mathcal{A}^\times$.

So the claim is that the fundamental groupoid acts in this way. Let me say a few words. More generally, the package "up to inner automorphisms," let me,

- $\mathcal{A}^\bullet$ is a graded algebra, $\nabla \in \mathrm{Der}^{+1}(\mathcal{A}^\bullet)$, and $\nabla^2 = \mathrm{ad}(R)$ for $R \in \mathcal{A}^2$ and $\nabla(R) = 0$, this is due to Positselski, a curved dga, and then everything I said, $\mathfrak{g}$ acts on $\mathcal{A}^\bullet$ up to inner automorphisms, and $T_g \circ \nabla \circ T_g^{-1} = \nabla + \mathrm{ad}(\beta(g))$ with $\beta(g)$ in $\mathcal{A}^1$. Again there are natural consistency conditions.

We have $\pi_1(X) = \mathfrak{g}$ acting on $(\mathcal{A}^\bullet, \nabla_{\mathcal{A}})$ up to inner automorphisms, and our objects are $(\mathcal{V}^\bullet, \nabla_{\mathcal{V}})$ with a compatible action of $\mathfrak{g}$ and $\mathcal{A}^\bullet$.

Those are my objects, and for them, given two objects $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$, let me just view them as $C^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$, the standard bar complex for $\mathrm{Ext}^\bullet_{\mathcal{A}}(\mathcal{V}, \mathcal{W})$. Then from the basics of homological algebra, you know that inner automorphisms act trivially on Ext functors. Then the second key fact is that $\pi_1(X)$, well, A_∞-acts on $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$, the discrepancies, it's not multiplicative, but the discrepancy is an inner automorphism which is trivial, there are higher homotopies. I have to stop here, finally we've seen some higher structure, let me conclude by saying that we can get objects from Lagrangian submanifolds in X, you get such a thing, but the morphisms give an A_∞-local system of $\mathbb{K}$-modules. I should mention that you probably get some higher category enriched with these, there is more subtle structure, and when it comes to higher structures, well, I have to stop and we're not sure how to say it correctly.

6. Dmitry Tamarkin: On the microlocal category

Let me thank the organizers for their kind invitation. So of course my talk should have som relation to Boris Tsygan's talk, but this relation still has to be studied. I probably won't make any statement about equivalence or functor or anything, but well roughly speaking one can make aparallel with Riemann–Hilbert correspondence. If Boris Tsygan's talk comes from D-modules and deformation quantization, I'll come from the other side, using sheaves of vector spaces, mostly constructible. If there is a relation, it should be a generalization of the Riemann–Hilbert correspondence.

The goal is the same, to construct a differential graded example as close to the Fukaya category as possible. I can still conjecture that the Fukaya category should sit in what I'm going to introduce as a full subcategory (those objects whose support is a Lagrangian). My construction is more complicated so I won't be able to go into all details. Maybe I'll focus on those aspects that are related to this conference. I don't know how high my structures are but at least higher than some other structures.

As in Boris Tsygan's talk, I'll start with M compact and symplectic. I'll have a more technical assumption, I want a prequantization bundle, so I'll assume that $[\omega]$ is an an *integral* cohomology class, maybe up to 2π, so we can realize it as the Chern class of a bundle L over M. My first step is to consider a non-compact but simpler example specifically T^*X with its canonical symplectic structure. and then we'll associate with it a category. Well, so, here the idea, we can try to use the Kashiwara–Schapira theory of sheaves (of vector spaces or Abelian groups) on X. Of course it's better to take the derived category of such, $D(X,\mathbb{Z})$, and then Kashiwara–Schapira teach us that we can get a conical subset $SS\mathcal{F}$ of T^*X, and if $\mathcal{F}$ is constructible this will be Lagrangian and in general it will be coisotropic.

The objects live over subsets of the cotangent bundle, this is the whole point of microlocalization. The difference with the Fukaya category is that here you only get conical Lagrangians. You need to modify it to get non-conical Lagrangians as well. This can be done by a standard trick of homogenezation. If I have something non-homogeneous, I can add a variable and make it homogeneous. [Picture]

You need to add extra dimensions, so you consider $D(X\times\mathbb{R},\mathbb{Z})$. Then the singular support will be a conic subset in $T^*(X\times\mathbb{R})$ and if I have something non-conic in T^*X I can convert it to a conic subset here, and do this procedure, writing it algebraically. I'll refer to a point in $T^*(X\times\mathbb{R})$ as (x,ω,t,k), where x is the point on the base, ω on the fiber, t in the $\mathbb{R}$ base, and k in the cotangent fiber $T^*_t\mathbb{R}$, so both t and k are just numbers. Then

$$\mathrm{cone}(L) = \{(x,\omega,t,k) | k>0, (x,\frac{\omega}{k})\in L\}$$

Since all conifications live in the positive fiber plane, we'll modify $\mathcal{D}(X\times\mathbb{R},\mathbb{Z})$ by quotienting by all things whose singular support is in the complement

$$T^*_{\le 0}(X\times\mathbb{R}) = \{(x,\omega,t,k)|k\le 0\}$$

and this is what I usually call $D_{>0}(X\times\mathbb{R})$, it turns out that this quotient is nice so the embedding has a right and left adjoint, so this is a left orthogonal complement, and then objects here can be specified by a simple formula. All objects are generated by sheaves, well, if f is lower continuous, then $\{(x,t)|t\ge f(x)\}$ is in the left orthogonal complement, and this is what everything will be built from.

This is more or less what happens in the cotangent bundle. By further quotienting, you can build a category supported on any open set. If U is open, then $\mathrm{sh}_U(X)$, you can build this, and from the construction you can see it's not clear if it uses the structure of the embedding of U into T^*X. To get to a general symplectic manifold we need to see that this only depends on U or [unintelligible], and then you can try to glue them which has further issues.

So first of all, we can consider simply an open ball B_R in $T^*\mathbb{R}^n$. If we believe that our quotient only depends on the symplectic structure of B_R, then I should have a symplectic group action on my category. I'll discuss only linear symplectomorphisms. The rest can be done by standard tricks, Alexander tricks and so on, that let you reduce arbitrary to linear symplectomorphisms. The hard part is the action of $Sp(2n)$ on the category. Then the next step is to glue. Somehow this reduces to a deformation problem, you have a complete local ring like formal series, and you have your construction [unintelligible]the maximal ideal and you want to lift it. So at least I hope to discuss the symplectic group action.

Also I should say, you don't have to do it for the ball, you can do it for the whole cotangent bundle. So the next question will be how the symplectic group acts on the category associated to $\mathbb{R}^n$. Part of the problem is to choose a differential graded model for your sheaf category. I'll use recent work of Jacob Lurie, only a tiny piece of this paper where he discusses topological obstructions to the Fukaya category, and we'll get exactly the same obstructions that he had, morally. There is also work in progress of D. Treumann and J. Xin, this is not published but since Xin is my neighbor and her office is next to mine, I know about it. For my approach I only need the differential graded part of it, but it's interesting to do it over spectra. So far everything goes smoothly if you use spectra as your ground category.

How do I define the $Sp(2n)$ action? Consider the category of sheaves on the product $Sp(2n) \times \mathbb{R}^n \times \mathbb{R}^n$, any action should be given by a kernel, and then any action will be given by convolution on $\mathbb{R}^n$. This has a monoidal structure using composition of kernels and convolution along the group. This has a symmetric structure, and this $\mathcal{F}$ will be a monoid with respect to this monoidal structure. By simple considerations, you can tell a priori what should be the singular support of $\mathcal{F}$. You can find a Lagrangian in $T^*(Sp(2n) \times \mathbb{R}^n \times \mathbb{R}^n) \times T^*\mathbb{R}$, and then we can switch to only looking at things supported in this Lagrangian L. This is rigid; for instance if this is the zero section this tells you your sheaves are local systems. You can assign initial conditions, for example if you specify [unintelligible]then you expect to get the identity kernel, so you can do stalks as well.

It turns out that what you can do, it's hard to solve the problem simultaneously over $Sp(2n)$ but if I restrict to a contractible open in $Sp(2n)$ then I can see that the space of solutions is the same as the category of spectra, so then in general we'll multiply by a category of spectra. So we define a certain locally constant sheaf of categories on $Sp(2n)$ whose fiber is just the category of spectra.

Then of course there is heavy artillery that classifies such things. There is a category of automorphisms of the category of spectra $\mathrm{Pic}(\mathbb{S})$, this is an infinity groupoid in the category of spaces, and you can take the classifying space, and the construction should lead to a map $\mathbb{S}p(2n) \to B\,\mathrm{Pic}(\mathbb{S})$. So the category we defined should have a monoidal structure (because things stay in L) and this means we should have a delooping, so I should get a map of topological spaces

$$B\mathbb{S}p(2n) \to B^2\,\mathrm{Pic}(\mathbb{S}).$$

It would be great to be able to define this directly. You find a map with the same target and source in Jacob Lurie's paper, and likewise in Treumann–Xin, and so they tell us, I can formulate a conjecture on what it should be, so first of all, we know that the symplectic group is homotopy equivalent to the unitary group, and then Bott periodicity, then $BU(n) \cong B^2(\mathbb{Z} \times BU)$,a nd then they say this is the j homomorphism $\mathbb{Z} \times BU \to \mathrm{Pic}(\mathbb{S})$. I'll leave this at that because I just started thinking about this, but let me say, there's some bad news that this map is nontrivial, and so my sheaf of categories is nontrivial and I can never find my solution $\mathcal{F}$. So for me, I'm not working with spectra but in a dg setting, I have Picard of type [unintelligible]. I can also work with the universal cover of $Sp(2n)$, and then I have no room for such a map. So switching to dg categories the corresponding map is automatically trivial. Then you can think about what is happening in spectra.

By doing certain tricks, without any higher structure you can solve the following problem. Suppose you have a symplectic embedding F, then you'd like a functor between the corresponding categories. We need to be careful because we lifted to the universal cover of $Sp(2n)$, and the functor is not defined canonically from F, to construct the functor unambiguously you need additional information, such as $dF \in Sp(2n)$, but you need to provide a lifting to $\widetilde{Sp(2n)}$, you could do this in one point or all simultaneously, this doesn't matter. But if your base is not simply connected this might not be possible globally.

This gives standard gluing procedures that we need. It turns out that it's not sufficient to solve the problem. You really need to, you need one more step. Why this knowledge of how to embed one ball into another is insufficient to build a global category on the symplectic ball is because, well, you have U in $T^*\mathbb{R}^n$, and you can associate $\mathrm{sh}_U(\mathbb{R}^n)$ by quotienting the things supported away from U. This is only a *pre-sheaf* which (somewhat unusually) is good news because if it were a sheaf, you could not hope for this to be the Fukaya category; that failure of sheafiness is what leads to instantons and so on in the Fukaya category. While it's not a sheaf, if the intersection points are far away, the instantons, the holomorphic disks, should have large area. So if I quotient by all disks with large area, then my interaction is confined to a small region in space. Then I can hope to build my category from just balls and embeddings of balls.

But to do something for symplectic manifolds, we need to find out how to do this reduction. For the Fukaya category, you cut off large balls or take a quotient of the Novikov ring, so how do we mimic this in our setting? Our $\mathcal{F}$ is in $D_{>0}(X \times \mathbb{R})$ and we have translations T_a in the $\mathbb{R}$ direction, and so I have an endofunctor here and now, let me remind that a typical object is $\mathbb{Z}_{\{t\geq f\}}$ and if I translate this, it becames $\mathbb{Z}_{\{t\geq f+a\}}$ and this is smaller so I get a canonical map from a larger to smaller closed set. Then for any positive shift, I get a natural map $\mathrm{Id} \to T_a$ and then I can modify my category structure (I want to define the classical limit), I want to modify the Hom of my category. We define a new category, the "classical reduction" or something like that, defined as follows. We let $\epsilon > 0$ and I want to kill areas larger than ϵ. My objects are the same as before, $\mathrm{sh}(X)$, but $\hom_\epsilon(\mathcal{F}, \mathcal{G}) = \mathrm{cone}(\hom(\mathcal{F}, T_{-\epsilon}\mathcal{G}) \to \hom(\mathcal{F}, \mathcal{G}))$. This ceases to be a triangulated category, it has more maps and you need cones of these newly arisen maps. Morally it's a quotient.

Then the point is that you still need to make rigorous, under this reduction you can indeed globalize our local data for any non-even-necessarily closed manifold, but you need a quantization bundle for certain reasons. If I have sheafs for each set in my cover, you can try to glue together, maybe it looks different if you do it technically, but this is what it is basically. If in the Fukaya category you kill instantons it's not interesting so eventually you want to include all the instantons. This requires a quantization that relies on higher structures. The only hope is to find a general theorem to provide the existence of a quantization, vanishing of a homology or something verifiable. Maybe it's not that motivated if I just formulate the theorem, maybe I'll try to do it in a meaningful way. Maybe I'll abstract from these details and try to formulate it generally.

As usual in quantization theory, you should have a local ring, I also forgot to say, crucially, everything so far can be done over the integers, but here the quantization has torsion obstructions and so you need to switch to rational numbers. This was reassuring because in the Fukaya category, your moduli spaces are orbifolds and fundamental classes have rational coefficients because you need to quotient out by finite groups.

So we work with $\Lambda = \mathbb{Q}[[q]]$ and all objects have additional structure, a grading where q is in degree -1 and then as usual, suppose I have an object over $\mathbb{Q}$, I should get an object over the local ring, and it turns out that all in all if I try to formulate the quantization, I have a monoidal category M over Λ and an alegbra A in the reduced category $M/(q)$, and the question is to lift it to M. The A is the solution to the gluing problem modulo ϵ and you want to lift it to formal series, and then I reformulate it in, I need to first laxify everything, and so my lax version for a monoidal category will be an operad, a colored symmetric operad. Any monoidal category M, if I have objects $A_1, \ldots, A_n$, then I can always look at $\hom(A_1 \otimes \cdots \otimes A_n, A)$, I get a symmetric colored operad colored by objects of M. I think it suffices to restrict to the A that I already constructed, lifted to the quantum level. So then I have $\mathcal{O}$ over Λ and I have the operad of associative algebras mapping to $\mathcal{O}/q$ and then I want to lift this to the quantum level.

Like in physics, it's useful to add some Batalin–Vilkovisky flavor to this, and the category has additional structure, of a trace, you can categorify a trace, from algebra with trace you can define a monoidal category with trace $TR : M \to \Lambda-$ mod (the ground category) and it should be a part of the structure that the trace should be cyclicly invariant in the appropriate sense. The operadic analog is what I called a *circular operad*, you have traces $A_1 \otimes \cdots A_n$, you can get something with no outputs, only inputs. You can do this covariantly or contravariantly, and then you have insertion maps where you can plug into each cog of this cogged wheel, you have your usual operadic, this is part of modular operads. Then you can define something, our algebra has a trace now, and for our algebra structure, you have a map from $TR(A)$ to the unit of the ground category, cyclicly invariant, and that's where you have a general theorem which provides for such a quantization.

Let me sketch where this comes from? What you can do with this cyclic operad is what I call the c_1-localization trick, you have the non-cyclic part of the operad, $\mathcal{O}_{\text{noncyc}}$, and the remaining structure is linear over the wheels, the remaining structure is a functor from a certain category whose objects are wheels, Boris calls it $\mathcal{Y}(\mathcal{O}_{\text{noncyc}})$, and you have it this extra bid, a module over $\mathcal{Y}(\mathcal{O}_{\text{noncyc}})$. If I have two fixed objects, I have product $\mathcal{Y}(\mathcal{O}_{\text{noncyc}})(m,n) \otimes N(m,n)$, tensored over cyclic objects, and now if I take the constant cyclic object, this $\underline{k}$ will act neutrally, and one endomorphisms I get is the map of degree 2 that you can call different ways (say, the Chern class). Then given an object you can try to invert c_1 and from the original operad you can get a new one $\mathcal{O}^{\text{loc}}_{\text{cyc}}$, and all the nontrivial action of the noncyclic part vanishes. So this boils down to something very simple, and that's what allows, I'm running out of time, so I'll just stop here, this is the major trick. You solve your problem here in $\mathcal{O}^{\text{loc}}_{\text{cyc}}$ and if you compare deformation complexes you see they are retracts of each other.

Thank you for your patience.

7. Yuri Manin: Interaction of computability theory with structures, categories, and higher structures

Thank you. I will first introduce the meaning, a minimalistic definition of the object I want to think about. It is a category $\mathcal{C}$ whose objects are the positive integers $\mathbb{Z}^+$ and also all initial intervals $[n] = \{1, \dots, n\}$. The morphisms, if you have a couple of objects, then $\mathcal{C}(X, Y)$ are (partially) computable maps from X to Y. Right away I must say that I will define this more precisely later on in several different ways. There is a challenge, to find homotopy alegbra relevant to $\mathcal{C}$. We know know in our era of brave new rings that all integers are embedded into ring spectra and spherical ring spectra, and there are interesting non-trivial homotopy questions behind all of it. So far, so far as I know, no one seriously tried to do something like this fo maps.

So first of all, one remark, that as I said, morphisms are not necessarily everywhere defined computable functions. So $f : X \to Y$ is actually a pair $(f, D(f) \subset X)$ and $f : D(f) \to Y$. So for composition of f and g, we take $f^{-1}(D(g) \cap \mathrm{im} f)$ as $D(g \circ f)$. It's not a very convenient category, so people deal with not everywhere defined maps by adding to each set one infinite or abstract element. Instead of dealing with the category of sets with the usual maps, one can define in this way a category of sets with partially defined morphisms, and one can define the category which I will call PSets where objects are sets endowed with a specific element $*$ and morphisms are restricted by the fact that $*_X \to *_Y$, so they are everywhere defined but with one restriction. There is a functor from ParSets to PSets by adding an element and whenever a morphism is not defined send it to the infinity point. So we can look at this embedding, everything will be everywhere defined. Here one more useful construction emerges before passing to it in full generality, I will give one definition of partially computable functions. This will not fit right away into this framework. Here we will consider a finite alphabet A, and words in A, $W(A)$, these are finite words. Then Markov's algorithms or semicomputable functions are defined in the following way. You produce an "obviously computable" fixed complete ordering of words $\mathbb{Z}^+ \xrightarrow{\sim}$, for example a dictionary, you first consider all words of length 1, of length 2, and so on, and inside by alphabetical ordering.

This is the list, and then one algorithm is numbered by a finite ordered list of pairs of finite words, an element P in $W(W^2(A))$. Then the algorithm, if you take an input word $w \in W(A)$, then in order to apply the algorithm P, you look at the finite subwords of w, and look at the first list of P, and look at the first word in the list? Is it somewhere in w as a subword? Take the first word and replace it with the second. It may happen that after a finite number of steps you get a word with no appearances of any word in the first half of P. Maybe for simplicity you have one more word associated to the word stop, then you get a definition. It can happen that you don't find a word, then it is outside the domain of definition. It might happen that you go to infinity, then you're outside the domain of definition.

In a sense, Markov algorithms constitute one of multiple equivalent definitions of computability, and you cannot do better. Turing machines do the same, partial recursive functions do the same, you should invent how to translate one to the other, but all that were invented do the same. Let me formulate it, a maximalistic definition of the category $\mathcal{C}$, objects are now sets X together with one to one bijections to $\mathbb{Z}^+$ or one of the integer sets $\{1, \dots, n\}$, and this bijection must be "intuitively algorithmicly computable" along with its inverse. I leave a way for people to come up with new algorithms. Then morphisms are algorithmically computable on the respective numerotations. I will formulate here what I call "categorical Church's thesis," that any two categories defined in this way are equivalent. I call it an experimental fact in the world of ideas, it's not a theorem, because you are including intuitive notions. Whoever has managed to formalize the notion of algorithm has always landed in the same universe. This lets me work in whatever version I want to do now.

On the other hand, there exists now additional structure(s), namely this category admits pretty obvious monoidal structures. There is a direct product $X \times Y$ and $X \sqcup Y$ from the category ParSets, of course. What i want to say now is that the problem of extending numerotation to these, whatever way you do this, they are all equivalent to one another. As I said hte main challenge that I want to suggest is how to introduce homotopical structures into this categorical context.

What is the problem? The problem is that the set of morphisms $\mathcal{C}(X,Y)$ when at least one object is infinite, this is not itself an object of the category. There is not an internal Hom in this category. What is true is that for any X and Y there is an object of $\mathcal{C}$, say $P_{X,Y}$ and a morphism, again in $\mathcal{C}$, from $P_{X,Y} \times X$ to Y. You should imagine P as the set of descriptions of programs from X to Y. Therefore for p in $P_{X,Y}$, you can define a set theoretical map $\bar{p} : X \to Y$ such that $x \mapsto \bar{p}(x) = \mathrm{ev}(p \times x)$.

The relationship between morphisms and objects is not straightforward at all. You have $\mathcal{C}(X,Y)$ and also $P(X,Y)$, and in particular there are versal maps because there may be very stupid programs, but there are programs that produce all maps. The trouble is that you would like them if possible, you want P to be as close to $\mathcal{C}$ as possible. The question is for p and q, when does $\bar{p} = \bar{q}$ does not have a computable answer. I want to tell you a lot of interesting mathematical problems are of this kind. I'll start with an old one, Lagrange's theorem about the number of representations of an integer as a sum of two squares, and it's known that this number (assume N is odd; the even case is also easy) is $8\sum_{d|N} d$. This is a nontrivial number theoretical identity, and nowadays one says the left and right side are Fourier coefficients of the same [unintelligible]function. This of course is not an elementary proof. Of course, can you prove this in an elementary way? This means, the left hand side, a short description, and the right hand side, also a program for calculation, why do they produce all the one and the same answer? The proof would be a third program that would be, if you rewrite it in one way, you get one, and in the other way you get the other. The answer is known in this case. Let me give you a more recent (and unknown) example, about Kontsevich–Zagier periods.

So periods are complex numbers, an imaginary part and a real part, both written as an integral over a locally closed subset of $\mathcal{P}^n(\mathbb{R})$, and the forms you integrate are products of rational functions of coordinates with coefficients in $\mathbb{Q}$ multiplied by some volume form, in an obvious way also rational over $\mathbb{Q}$. These are not integers or words, but generally transcendental numbers, and the conjecture of Kontsevich and Zagier says that if you have two integrals like that, then they produce one and the same number if and only if you can pass from one such integral to the other by doing a short list of standard identities between integrals, Stokes' formula, variable change, things like that.

When I first read it in the well-known paper of Kontsevich–Zagier, I was incredulous, it's of the same type, you want in an elementary way to pass between one way and another of the same map.

In this general categorical setting, when you ask about this as a question about $P(x, y)$, this problem becames algorithmically unsolveable. So whatever tricks you invent to show that to pass from one to the other you calculate the same function, you'll never be able to understand this completely.

The conjecture is still unproved. The strongest argument for the conjecture is that in some sense it follows from Grothendieck's standard conjectures. I'm wondering whether I should consider what I'm saying as an argument against the standard conjectures.

There is one more detail that I failed to mention before. Now the two real numbers are not rational or integer, so in what sense am I speaking about the integral expressions as programs? What are they calculating? People who are doing numerical analysis, they have such integrals which encode computable real numbers where a computable real number is a number so that there exist an algorithm so that for any n, it calculates some amount of digits of that number which is distance no more than $\frac{1}{n}$ to your number.

This is not the most natural definition of computability. It might happen that what happens is that you can calculate any number of difficults, but you're ignoring how close you come to your number. Maybe you get 10 digits, and the 11th digit comes much later with a correction to the 10th one.

More precisely, you're approximating in a computable way not just objects of $\mathcal{C}$ but also some morphisms in $\mathcal{C}$. You have a morphism from $\mathbb{Z}^+$ to the space of digits. So when you have chosen a program, and you want to calculate a composition, $P(X, Y) \times Y'(Y, Z) \to P''(X, Z)$, you essentially deal with a problem as described, you want objects constructible so they produce programs. In the best case, this produces only a commutative diagram of partial sets which imitates the usual product but it's not computable in the good sense of the word. We are kind of bound to go up in this categorical intuition. You get not maps but programs to calculate maps. We must go one stage up and consider elements of programs programming this composition.

It is clear that higher structures are there by necessity, but nobody yet studied these seriously. I'm presenting you with a challenge. Now when I was thinking about this, I was trying ot look at two homotopical structures, one a model structure on such a category, and the other. The other is an imitation of spectra on this category. As I said, I did not come up with any definitie definition, but I'll just remind you that in order to define a model structure, we must at least have sufficiently many commutative diagrams in our categor so that for any commutative square you have some lifts. Using versal programming you can get some of these lifts, but I was not able to show that they satisfy the required compatibilitity.

Another challenge, another attempt to approach this challenge about spectra was motivated by the fact that if one considers the category of PSets, and then one asks how we transpose direct product and disjoint sum, then we find out that this product $X \times Y/\sim$ becaomes $(X, *_X) \times (Y, *_Y)/(*_X \times Y) \times (X, *_Y)$. This is of course a very well-knonwn elementary construction at the beginning of homotopy theory. So perhaps there is a natural analog of suspension of an object in $\mathcal{C}(X)$, just a product in this category of $Z_+ \times X$.

What about Ω?? I never suggested my Ω. The description of the algorithm contains Ω. You input something, and if you stop at stop you get the final thing. The microdescription of algorithm has in addition a macro description, something like a Feynman category. That's so to speak, global, you consider graphs whose output are previously constructed and when the output is the input of the next thing.

When there is a real understanding, the model structure, suspension, loop structure should be joined together into the same package.

8. Nicoló Sibilla: a categorification of the Chern character

So I want to thank the organizers for inviting me to speak, and I'll be talking about joint work with Mark Hoyois and Sarah Scherotzke.

Let me start with an introduction to character theory. We're probably either thinking about X a manifold or algebraic variety and then a Chern character from complex vector bundles on X to $H^*_{dR}(X)$, which, to make things super explicit, sends a line bundle L to the exponential of the first Chern class, $\sum \frac{c_1(L)^m}{m!}$.

The other example, if G is a finite group, then we go from complex finite dimensional G-representations to $\mathcal{C}(G) = \mathcal{O}(G/G)$, and these two stories are both aspects of a very general picture I'll sketch next with the help of homological algebra and a little bit of derived algebraic geometry.

Let $X/\mathbf{k}$ be a scheme, or a stack, then the Chern character is packaged in the following diagram:

$$\begin{tikzcd} i_0 \operatorname{Perf}(X) \arrow[r, "\mathrm{Tr}"] \arrow[d] \arrow[dr, dotted] & HH_*(X) \ni HH_*(\operatorname{Perf}(X)) \\ k_*(X) \arrow[r, "\mathrm{ch}"'] & HH_*(X)^{S^1} \arrow[u] \end{tikzcd}$$

This is a very classical story. I'll write down three features that will be important and then reformulate this in a different away.

(1) The Chern character is multiplicative,
(2) additive, and
(3) admits an S^1-equivariant refinement.

This last is Connes and Tsygan, anyway, let me reformulate this. We'll recover the character theory of a finite group, I should say, by considreing $X = [*/G]$.

In order to reformulate the square, I want to consider the Hochschild homology itself as a trace. I want to introduce some definitions that are familiar to most of us about fully dualizable objects.

Definition 8.1. Let C be a symmetric monoidal category, sometimes an $(\infty, 1)$-category with unit 1_C, and X an object of C, then we say X is *dualizable* if there is an object $X^\vee$ in C and maps $1_C \xrightarrow{\mathrm{coev}} X \otimes X^\vee \xrightarrow{\mathrm{ev}} 1_C$ such that tensoring with X in the appropriate way we recover the identity of X

$$\begin{tikzcd} X \arrow[r, "\mathrm{coev}_\otimes 1_X"] \arrow[rr, bend right, "1_X"'] & X \otimes X^\vee \otimes X \arrow[r, "1_X \otimes \mathrm{ev}"] & X \end{tikzcd}$$

So if X is dualizable and $f : X \to X$ is an endomorphism then $\mathrm{Tr}(f) \in \mathrm{Hom}(1_C, 1_C) =: \Omega(C)$

$$\begin{tikzcd} 1 \arrow[r] \arrow[rrr, bend right, "\mathrm{Tr}(f)"'] & X \otimes X^\vee \arrow[r, "f\otimes 1_{X^\vee}"] & X \otimes X^\vee \arrow[r] & 1_C. \end{tikzcd}$$

Then we define $\dim(X)$ as $\mathrm{Tr}(1_X)$ and maybe I should say that we recover ordinary dimension from this.

So now let me move into derived ∞-stacks over $\mathbf{k}$. This includes ordinary schemes and stacks, simplicial sets, and derived affine schemes over $\mathbf{k}$, commutative differential graded algebras over $\mathbf{k}$.

Now $\underline{\mathrm{Cor}}_k$ is the following symmetric monoidal $(\infty, 1)$-category, the objects are these guys, and then morphisms are correspondences

$$\begin{tikzcd} & T_1 \arrow[dl] \arrow[dr] & \\ X & & Y. \end{tikzcd}$$

Composition of a correspondence is just given by the derived fiber product.

$$\begin{tikzcd} & & T_1 \times_Y T_2 \arrow[dl] \arrow[dr] & & \\ & T_1 \arrow[dl] \arrow[dr] & & T_2 \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Something I learned from Ben-Zvi and Nadler is that in this category all objects are dualizable (and self-dual), and the correspondences are given by

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "A"] & & X \arrow[dl, "A"'] \arrow[dr] & \\ \mathrm{pt} & & X \times X & & \mathrm{pt} \end{tikzcd}$$

and now let's compute what this correspondence is by taking the derived fiber product.

What we obtain is that we have the following:

$$\begin{tikzcd} & & X \times_{X\times X} X \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr, "A"] & & X \arrow[dl, "A"'] \arrow[dr] & \\ \mathrm{pt} & & X \times X & & \mathrm{pt} \end{tikzcd}$$

So then the dimension of X is then the derived loop stack of X. Again, for the few people, the many people who haven't seen this before, let me tell you why this is the loop stack of X. This is $\mathcal{L}X = \mathrm{Maps}(S^1, X)$, and then this is $X \times_{X\times X} X$ because I can write S^1 as $\sqcup_{*\sqcup *} *$, this is a familiar observation from topology.

Then $\mathcal{L}X$ is the "universal trace" in some sense.

Now we'll define a sheaf theory from this category of correspondences.

Definition 8.2. A *sheaf theory* is a monoidal functor $\underline{\mathrm{Cor}}_{\mathbf{k}} \to \mathrm{dgCat}_{\mathbf{k}}$, and again for this we should assume some finiteness that I'll suppress on the stacks that we consider.

So the sheaf theory we want sends X to quasicoherent sheaves on X and sends the correspondence

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

to the map

$$\mathrm{QCoh}(X) \xrightarrow{q_* p^*} \mathrm{QCoh}(Y).$$

In setting this up, we can compute the trace of $\mathrm{QCoh}(X)$ in $\mathrm{dgCat}_{\mathbf{k}}$ in a very simple way.

The trace of $\mathrm{QCoh}(X)$ as a dualizable object (I'm being vague about technical details about dgCat) is given by global sections $\mathcal{O}(\mathcal{L}X)$ of the loop stack, which is clear immediately by considering the following diagram

$$\begin{tikzcd} & & \mathrm{QCoh}(X) \arrow[dl] \arrow[dr] & & \\ & \mathrm{QCoh}(X) \arrow[dl] \arrow[dr] & & \mathrm{QCoh}(X) \arrow[dl] \arrow[dr] & \\ \mathrm{QCoh}(\mathrm{pt}) & & \mathrm{QCoh}(X \times X) & & \mathrm{QCoh}(\mathrm{pt}). \end{tikzcd}$$

Now I'll give a different interpretation, armed with this, of the diagram defining my Chern character.

$$\begin{tikzcd} i_0 \,\mathrm{Perf}(X) \arrow[r] \arrow[d] \arrow[dr, dotted] & \mathcal{O}(\mathcal{L}X) \\ k_*(X) \arrow[r] & \mathcal{O}(\mathcal{L}X)^{S^1} \arrow[u] \end{tikzcd}$$

Due to Toën and Vezzosi, this is the same as the diagram above. Here ι_0 is the maximal subgroupoid.

The intuition behind the existence of the circle action, this is a beautiful interpretation of the Chern character by Toën and Vezzosi.

What are we doing here, we're pulling back a bundle to $\mathcal{L}X$ but with our definition of loops, every bundle is locally constant, so the monodromy gives a natural automorphism, and the map we get is the trace of that automorphism.

$$\begin{tikzcd} & p^*E \arrow[dr] & \\ E \arrow[dr] & & \mathcal{L}X \arrow[dl] \\ & X & \end{tikzcd}$$

so $E \mapsto \mathrm{Tr}(\mathrm{mon}(p^*E)) \in \mathcal{O}(\mathcal{L}X)$.

This should exist in a more general context, so I'm going to write down a categorification of the Chern character.

So instead of a functor to dgCat, we should be able to consider a functor to an $(\infty, 3)$-category of [unintelligible]. Let me write down a table.

ordinary	categorified
$\mathrm{Perf}(X)$, the category of perfect complexes on X	$\mathrm{Cat}^{\mathrm{sat}}(X)$, the category of dualizable quasicoherent sheaves of categories on X
$\mathcal{O}(\mathcal{L}X)$, functions on the loop space	$\mathrm{Perf}(\mathcal{L}X)$, perfect [unintelligible]on the loop space
$\mathcal{O}(\mathcal{L}X)^{S^1}$, negative cyclic homology	$\mathrm{Perf}(\mathcal{L}X)^{S^1}$
algebraic K-theory of X, $k_*(X)$	a category $\mathrm{NMot}(X)$ of noncommutative motives over X (I'll explain this in just a minute).

Our main theorem is this diagram in the categorified setup.

Theorem 8.1. *(H.–S.–S.)*

$$\begin{tikzcd} i_1\,\mathrm{Cat}^{\mathrm{sat}}(X) \arrow[r] \arrow[d] & \mathrm{Perf}(\mathcal{L}X) \\ \mathrm{NMot}^{\mathrm{sat}}(X) \arrow[r, "\mathrm{ch}"'] & \mathrm{Perf}(\mathcal{L}X) \arrow[u] \end{tikzcd}$$

Let me tell you some consequences of this diagram.

(1) The further restriction
$$i_0\,\mathrm{Cat}^{\mathrm{sat}}(X) \to k^{S^1}(\mathrm{Perf}(\mathcal{L}X)) \to \mathcal{O}(LLX)^{S^1\times S^1}$$
is the secondary Chern character of Toën and Vezzosi.

(2) the secondary K-theory
$$k_0^2(X) \to \pi_1\mathcal{O}(LLX)^{S^1\times S^1}$$
(introduced by Toën and previously independently by Bondal–Larsen–Lunts when $X = \mathrm{Spec}\ \mathbf{k}$), let me explain what this secondary K-theory looks like.

When X is the spectrum of a field, this is the free Abelian group on smooth and proper triangulated dg categories over $\mathbf{k}$, quotiented out by the relation that when we have $A \hookrightarrow B \twoheadrightarrow B/A$, we say that $[B] = [A] + [B/A]$, in complete parallel to the ordinary Grothendieck relation.

Another observation which is very vague is that according to Ben-Zvi and Nadler, $\mathrm{Perf}(\mathcal{L}X)^{S^1}$ should be related to a category of filtered D-modules over X, the Chern character seems to incarnate the functor that sends to a motive its variation of mixed Hodge structure.

In the last five minutes let me say a little bit about the category of non-commutative motives over X and why it deserves to be called the K-theory of this categorified thing.

So just for simplicity, let me assume that X is so-called 1-affine, which means that this category of quasicoherent sheaves of categories over X can be expressed more concretely as categories tensored over $\mathrm{QCoh}(X)$. Then we can characterize our non commutative motives by saying that any functor to a presentable stable $(\infty,1)$-category that commutes with filtered colimits and sends $A \hookrightarrow B \twoheadrightarrow C$ to $D\Phi(A) \to \Phi(B) \to \Phi(C)$

$$\begin{tikzcd}
i_1 \mathrm{Cat}^{\mathrm{sat}}(X) \arrow[r, "\phi"] \arrow[d, "{[unintelligible]}"'] & D \\
\mathrm{NMot}^{\mathrm{sat}}(X) \arrow[ur, dotted] &
\end{tikzcd}$$

So some version of this category was first constructed by Kontsevich as a certain triangulated envelope, when X is $\mathrm{Spec}(\mathbf{k})$, in the following manner, the objects are saturated dg categories and the morphisms from A to B are given by $k_*(A \otimes B^{op})$. This definition in Kontsevich becomes part of the theorem for us, and let me just conclude with [unintelligible].

Theorem 8.2. *(H.–S.–S.) If A and B are in $\mathrm{Cat}^{\mathrm{sat}}(X)$, then*

$$\mathrm{Mor}([unintelligible](A), [unintelligible](B)) \cong k_*(A \otimes_{\mathrm{Perf}(X)} B^{\mathrm{op}}).$$

and this [unintelligible]work of Tabuada and Blumberg–Gepner–Tabuada.

The multiplicativity we obtain is [unintelligible], but $\mathrm{QCoh}(X)$ has a lot of interesting structure, there's work in progress with Ralph in the case of point mod G, where we want to twist the multiplicative structure. I'll stop here.

9. Bertrand Toën: Motives and dg-categories

We heard in the last lecture, I'll use very similar ideas, and before starting, this is joint work in progress with Blanc, Robalo, and Vezzosi. I'll talk about things inspired by Blanc's thesis, about topological K-theory on a non-commutative space coming from dg categories, together with a Chern character in this context. I want to use the same kind of ideas in a more arithmetic situation. The goal is to study non-commutative spaces, which for me are dg categories in arithmetic situations, and I want to work over any base A, a discrete valuation ring in any characteristic.

We want cohomology theories for non-commutative spaces, ℓ-adic cohomology, and this will be a variation of Blanc's description of topological K-theory.

If you want a more specific motivation for what I will say, we want, for any dg category T over A, we want to construct ℓ-adic cohomology of T with $\mathbb{Q}_\ell$ coefficients, this is a $\mathbb{Q}_\ell$-adic complex over $\mathrm{Spec}(A)$, and we want trace formulas for these things. If I have an endomorphism of the dg category, I want the trace of the induced morphism on homology from invariants of T. I said this is in progress. We can prove a trace formula but we have some examples where we have to check our conditions are satisfied. A future application is to try to prove a conjecture called Bloch's conductor formula out of this formalism. This is an incarnation of a trace formula for a non-commutative space, and this is what I'll try to explain near the end of the talk.

(1) I'll spend some time talking about motivic BU-modules (others call this KGL but I'll prefer BU)
(2) I want to talk about ℓ-adic realizations, and then
(3) ℓ-adic realization of matrix factorization.

You'll see that we get pretty close to Bloch's conductor formula.

This conference is about higher structures, so I should tell you about the relation to higher structures. What are the aspects?

(1) First, it uses ∞-categories everywhere. It's not so surprising, if you are doing functorial homotopy things, you expect this, so that's okay.
(2) We need some base rings that are topological in nature and they will be E_2 rings. They come from algebraic geometry, not spectra, but still they are E_2 rings. We've heard about traces but what happens over an E_2 ring is a little more complicated. So there are also some aspects related to TQFT, how do I call them, is Claudia here? Ah, twisted, so for me, twisted, defects, boundaries, some kind of thing like this.

9.1. **Motivic BU-modules.** . So S is $\mathrm{Spec}(A)$ for A a commutative ring, which will be nice when needed, maybe Noetherian, I don't know, I won't try to give the most general statement.

Let me recall the stable homotopy theory of schemes. For me a stable homotopy theory of schemes over S is a symmetric monoidal ∞-functor from schemes over S to a target symmetric monoidal ∞-category, presentable, $(\mathcal{D}, \otimes)$, satisfying some conditions, such that

(1) $X \times \mathbb{A}^1 \to X$ gets inverted in $\mathcal{D}$, the image of this map through this functor is an equivalence in $\mathcal{D}$
(2) You have [unintelligible]descent, when you have a [unintelligible]square, it goes to a coCartesian diagram in $\mathcal{D}$, and
(3) $\mathbb{P}^1/\infty$ goes to a tensor-invertible object in $\mathcal{D}$.

It's a fact that there is a universal such theory given by Morel–Voevodsky from smooth schemes over S to $\mathcal{SH}_S$.

We heard a similar construction in the last talk, there is a non-commutative version, this was done by Tabuada (in a different way) and Robalo, who I'll follow, this is a slight modification of Tabuada.

I'll start with some dg categories over S, by taking perfect complexes on the scheme, and smooth goes to "finite type," the non-commutative smooth condition, not exactly, a little stronger than the smooth category,

$$\mathrm{Sm}/S \leftrightarrow (\mathrm{dgCat}^{\mathrm{ft}}/S)^0$$

Let's do the same, A, B, and C, the same construction as before, but with $(\mathrm{dgCat}^{\mathrm{ft}}/S)^{\mathrm{op}}$, and let me say what this means, so $\mathbb{A}^1$-invariance is easy. The descent is a little more tricky, you should think that if I have a quotient of a dg category by a compact object, it goes to a triangle, but then there are some technical details.

Then the third one is obvious, implied by the second because $\mathbb{P}^1$ fits in an exact sequence. So the last one is not necessary. The short exact sequences are like Verdier's quotient, right? These go to exact triangles.

So then you get this new thing, let's call it $\mathcal{SH}$ non-commutative, the universal such thing, $(\mathrm{dgCat}^{\mathrm{ft}}/S)^{\mathrm{op}} \to \mathcal{SH}_S^{\mathrm{nc}}$, and this is the dual of Tabuada's construction plus $\mathbb{A}^1$-invariance.

Now there is a natural symmetric monoidal ∞-functor, stable, that goes from $\mathcal{SH}_S \to \mathcal{SH}_S^{\mathrm{nc}}$ characterized by saying that it sends $S' \to S$ to perfect complexes on S'. I get this map, call it j, and now there is an important result here, which has to do with the adjoint of j, the heart of the construction of [unintelligible]of dg categories.

Theorem 9.1. *(Tabuada, Robalo)*

1. *j has a right adjoint $\ell : \mathcal{SH}_S^{\mathrm{nc}} \to \mathcal{SH}$ (now only lax monoidal)*
2. *$\ell(\mathbf{1})$ (automatically an E_∞-monoid) is equivalent to BU_S in $\mathcal{SH}_S$, where BU is an E_∞ motivic spectrum, let me focus on its K-theory, it presents algebraic homotopy invariant K-theory. This is usually called KGL.*

The important statement is the second point.

As a consequence I get a new monoidal adjunction between $BU_S - \mathrm{Mod}$ and $\mathcal{SH}_S^{\mathrm{nc}}$. We are almost done, let me give you the definition of the motivic BU-module associated to a category and then some examples and what the construction really does. Intuitively it's pretty clear what this is, I'll give it to you in a moment, but there are a lot of coherences to check. It's one of these things, if you see the statement you think it's obvious but you have to write a proof, it turns out to be a hundred page paper.

Definition 9.1. The motivic BU_S-module associetd to T, a dg category, and I don't need finite type because I can write anything as a filtered [unintelligible]of finite type ones, so I can just extend, is

$$M^T \coloneqq \ell(T^\vee)$$

where $T^\vee \in \mathcal{SH}_S^{\mathrm{nc}}$.

I'm going to define M^T from smooth schemes (op) to spectra, and this takes $\mathrm{Spec}(A' \to S)$ to $HK(T \otimes_A A^1)$, this is homotopy invariant K-theory, where H is just something functorial to make it homotopy invariant. This is a module over $HK(A')$, this is the intuition. It just comes with this extra structure that you need some results to be sure that it exists.

This has some properties.

So some basic facts,

- $T \mapsto M^T$ is a lax monoidal ∞ functor.
- if $X \xrightarrow{p} S$ is a flat scheme, maybe a little bit more, and let T be perfect complexes on X, then M^T is $p_* BU_X = BU_S^X$.
- $T \mapsto M^T$ sends short exact sequences to exact triangles, and finally

If we want to study algebraic K-theory of X, you can look at BU_S^X as a BU-module. So this is like a compatibility between the commutative and non-commutative.

9.2. **ℓ-adic realizations.** Now it's kind of easy, we have an object in a category of motives, an ℓ-adic realization exists for motives, we need to keep track of the BU-module structure. I can have exotic bases, like E_∞ algebras, so the realization should keep track of this base.

First of all by the universal property of $\mathcal{SH}_S$, whenever I have, well, there is an ℓ-adic realization functor r (here ℓ is invertible in A) from $\mathcal{SH}_S \to L(\mathrm{Set}, \mathbb{Q}_\ell) =$ ind−constructible $\mathbb{Q}_\ell$ complexes on Set fitting in

$$\begin{tikzcd} \mathcal{SH}_S \arrow[r, "r"] & L(\mathrm{Set}, \mathbb{Q}_\ell) \\ \mathrm{Sm}/S \arrow[u] \arrow[ur] & \end{tikzcd}$$

where the diagonal map sends $S' \xrightarrow{p} S$ to something called $p_\#(\mathbb{Q}_\ell)$, the $\mathbb{Q}_\ell$-homology of S' over S.

That's a symmetric monoidal stable ∞-functor. This satisfies descent because you have étale descent. It is $\mathbb{A}^1$-stable because [unintelligible]. Then by universal properties, you get this.

So r goes from BU_S-Mod to $r(BU_S)-$Mod. NOte that $r(BU_S) \cong \bigoplus \mathbb{Q}_\ell(i)[2i] =: \mathbb{Q}_\ell(\beta) = \mathbb{Q}_\ell[\beta, \beta^{-1}]$.

Okay, and so I can apply r to T and get a module here. This is not strictly two-periodic because when I shift by two I have to twist by 1. So these are Tate-twisted two-periodic objects, $E + E[2] = E(1)$.

Let me now add a base, an extension, and then I will [unintelligible]. Suppose now that B is an E_∞ A-algebra, in fact E_2 is enough. Suppose that T is a B-linear dg category. Then M^T is a M^B-module, so $r(T)$ is in $r(B)-$Mod I should say I'm saying r to mean also $r(M^T)$.

Let's define now a realization functor. You can check that the ℓ-adic [unintelligible], if you start with T being perfect complexes on a (maybe proper) scheme, then the ℓ-adic realization is the pushforward of [unintelligible], and this is, contrary to the other case, where I got BU^S, this is not easy, this uses a not-easy result by [unintelligible]in a base not a field. The comparison with ℓ-adic [unintelligible]of schemes is not an easy result but it is true for proper schemes.

9.3. **Matrix factorization.** Let me just state the results and then that's probably the end. Is it factorisations or factorizations? British versus American? Which is British [laughter]

Okay, so A is a discrete valuation ring, $\mathbf{k} = A/\pi$ is perfect, maybe I want it to be Hanselian, and $K = \mathrm{Fac}(A)$, and $X \xrightarrow{p} S$ is $\mathrm{Spec}(A)$, and p is proper (flat), X_K is smooth over K, and X is regular. So we set $T = MF(X, \pi)$, I'll give you a naive description of the objects, they are pairs E_0, E_1 of vector bundles on X with maps $E_0 \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_0$, and $\delta^2 = \times\pi$. I don't want this to be a category over A alone because this is 2-periodic. So this is $A[u, u^{-1}]$-linear, where u has degree 2, and this is very important extra data. This sounds stupid, not very deep, but this 2-periodic structure has a lot to say.

I can take vanishing cycles in the ℓ-adic cohomology of this map, I'll take the special fiber extended to the algebraic closure of K, and I want these to be invariant with respect to the Galois [unintelligible], $\mathcal{O}_p \in L(\bar{X}_0, \mathbb{Q}_\ell)$ with the action of G_K

Theorem 9.2. *(Blanc–Robalo–Vezzosi–T.) There exists an equivalence* $r(T) \xrightarrow{\cong} \mathbb{H}^\bullet(\bar{X}_0, \mathcal{O}_p[-1])^{hI}(\beta)$ *as* $r(A[u,u^{-1}])$*-modules,* $r(A[u,u^{-1}]) = \mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)(-1)[-1]$*.*
Here I *is inertia,* $\ker(G_k \to G_k)$ *(and* h *means homotopy invariant)*

This plus the trace formula is the road to Bloch's formula. I don't claim we know how to do it, but that's the road.

10. March 22: Peter Teichner: twisted field theories from factorization algebras

[Outline on slides]

10.1. **Factorization algebras.** This talk is about connecting two different ways of mathematizing quantum field theory Factorization algebras is a way of formalizing the observables, and the cool thing is that you start with a classical theory. So you start with a classical BV theory. We fixe a space-time, a d-maniold You have fields and antifields and there may be ghosts and antighosts and some classical action functional S on the space of all these things. Out of all of that you produce what is called the derived critical locus. In general this is a derived stack, but $\mathrm{Crit}^{\mathrm{der}}(M,S)$ is a cochain complex in the settings I have in mind. This models the derived critical locus of the classical field theory. I'll explain this in one example. This will be called $\mathrm{Obs}^{\mathrm{cl}}(M)$. Then there's this cool machine, this book on renormalization and effective field theory, this gives a way to give perturbative quantization into what they call the quantum observables. I'll show you this $\mathrm{Obs}^q(M)$ in one example. This factorization algebra axiomatizes the structure present here.

This is very vague so let me do one example carefully, and that's the free boson. In this book you find the quantization of Yang–Mills, several others, and this is a strength of the approach, you get mathematical descriptions of these games with Feynman diagrams that produce physics. So I should maybe say g is a geometry on d-manifolds needed to define all these data. So g could be just an orientation or a framing or something, then this is just a topological theory. It could be geometric, conformal, complex, so g is a very general notion, the only thing I need is that you can glue when you have two open sets. This is the key interest, the physically relevant theories are geometric. The spacetime for us will be Riemannian.

The fields for the boson will be C^∞ functions, and let me explain how you get to the chain complex. You write down the antifields, the fields and antifields is two copies of $C^\infty(M)$ in degree 0 and degree 1, and you put the Laplacian plus the mass operator $\Delta + m^2$ as a differential between them, and this is the derived version of the classical solutions. Let me call it $\mathcal{E}(M)$ You don't just look at the kernel but the whole complex. In this example, that's what derived means.

Now classical observables $\mathrm{Obs}^{\mathrm{cl}}(M)$ are functions on the classical solutions. Here we have functions on the derived version. This is a linear space so you can talk about polynomial functions. We're taking $\mathrm{Sym}(\mathcal{E}(M)^\vee)$. So we get distributions. This is $C^\infty(M)$ with its Frechet topology. There is a grading here. Sym^2, for example, is quadratic distributions, and maybe we should also take the completion, formal power series.

There's an important technical point that these don't multiply, so really you take the smeared distributions, you take $\widehat{\mathrm{Sym}}(\mathcal{E}_{\mathrm{CS}}(M))$, these are smeared distributions in the symmetric algebra. Maybe I'll spell this out in terms of the two blocks. These are polyvector fields on $C^\infty_{\mathrm{CS}}(M)$.

It turns out that in this chani complex, the classical action of a field is the usual classical action, so we're integrating

$$S(\varphi) = \int_M (\Delta + m^2)\varphi \cdot \varphi \,\mathrm{vol}_M$$

There's a bunch of things to write down that I don't have time to talk about, this is symplectic (shifted symplectic) so functions on it has a Poisson bracket, and the differential is $\{S, \quad\}$.

So in words, think of the chain complex as a trivial Lie algebra, take a sort of Heisenberg extension, and then take Chevalley–Eilenberg. So this is functorial. If you add φ^4 terms, if there are symmetries, you'll have obstructions for quantizing. Your S satisfies the classical master equation but there will be a quantum master equation you have to satisfy. As usual, obstruction theory, if you find one deformation you're finding more.

So they show, Costello–Gwilliam, that this stuff satisfies the axioms of a factorization algebra. Before I show you them, this thing where I start with M and write down the quantum or classical observables satisfying some natural properties, that

(1) you can always extend observables by 0 and
(2) you can compose observables on disjoint spacetimes.

One difference to quantizations that you may have seen, the classical observables obviously form a commutative differential graded algebra. The quantum observables don't form an algebra. You can't multiply observables (because of the uncertainty principle), but you can do it if the regions are disjoint. What's being deformed is not the product, what's being deformed is the differential. As a graded vector space, it turns out at least in this case, the quantum observables are $\mathrm{Obs}^{\mathrm{cl}}(M)[[\hbar]]$, but the differential is deformed.

Then out comes this mathematical definition.

Definition 10.1. A *geometric factorization algebra* is a symmetric monoidal functor $F : (g-\mathrm{Man}, \amalg) \to (\mathrm{Ch}, \otimes)$ such that

(1) a (Weiss)-cosheaf condition is satisfied
(2) $F(M_1) \otimes F(M_2) \xrightarrow{\sim} F(M_1 \amalg M_2)$

Here my category of manifolds is open manifolds with isometric embeddings. The chain complexes are also interesting with respect to some topological considerations, that's in the book.

So $F(M)$ is something like classical or quantum observables. Functoriality is extension, and then being symmetric monoidal means that you can multiply on disjoint union but we want to say that this is really the same.

I won't write the cosheaf condition explicitly, but if you know them on arbitrary small sets, then you can figure them out on big sets, this is a local to global condition.

If you know nothing about BV theory or field theory or something, this is a completely mathematical definition, and if you've ever worked with these things, this is not a very hard definition.

Let me now remind you of a functorial field theory. I think Segal suggested this when listening to a talk of Witten, and Atiyah picked it up and many others. I want to do the geometric case, these are not just TQFTs but geometric field theories. This is a (symmetric monoidal) functor from a bordism category to chain complexes. In the original version we have vector spaces but now we have chain complexes. You take a space-slice Y^{d-1} and you associate to it a state space of the quantum system on Y, and you take a time evolution, what mathematicians call a bordism, and you think of it as a bordism, $\Sigma^d : Y_{\text{in}} \to Y_{\text{out}}$, and this is supposed to give the time evolution of the state space.

This is very rough, and I should say where the geometry comes in. This is strange, that you have the codimension one thing, if this is a complex thing, say, the codimension one manifold doesn't really make sense.

The other thing, imagine your favorite theory, if g is conformal and $d = 2$ you get a conformal theory, you get a vector space for a circle and for a surface a linear map, and for a closed surface, a torus say, you get a function from the empty set to itself, and this is a function on the moduli space. In many examples you get not just a modular function but a modular form.

So I want to talk about a twisted version, where you have a T functor into dgCat with $\boxtimes$ instead of into chains. If I have a theory that's supposed to give a modular form, then here I get, for a surface a section of a line bundle over the moduli space. So the way you formalize a lot of the anomalies is with this twisted formalism. So then I should say a symmetric monoidal bifunctor T and a bitransformation E from $\mathbb{1}$ to T. The unit, for each circle gives the category of chain complexes. The "bi" of bifunctor, this is, the objects are the space slices. The one-morphisms are the bordisms, and the two-morphisms are isometries. I'll explain this carefully rel boundary. The bordisms are in a natural way a 2-category because you have manifolds, bordisms, and isometries. If I let T be **1** itself, then a natural transformation is just a functor into chain complexes because Ch doesn't have interesting 2-morphisms. So this is really a generalization of what we had before.

Let me make some of this more precise with my slides.

[slide section]

11. Chris Schommer-Pries: Higher Categorical Structures from 3D Topological Field Theories

Thank you, so my main subject are extended field theories. When I was a young topology grad students and I was first introduced to field theories, one motivation was manifold invariants, from closed manifolds. They had better properties than random manifold invariants, you could compute them by slicing into pieces.

We saw in Peter's talk how the compositional rule about functors (composites of cobordisms) is one way to express that locality of the invariants that you get. I'll talk about extended invariants, you'll have a higher category of cobordisms. For example you'll have something like $\mathrm{Bord}_{(0,1,2)}$, a symmetric monoidal 2-category where the objects are 0-manifolds, the morphisms (1-morphisms) will be bordisms, for example a pair of elbows between two points and two points. Then the 2-morphisms will be diffeomorphisms between bordisms [picture]. An extended field theory will then be a functor from this higher category to a target higher category. An *extended topological field theory* is a symmetric monoidal functor. You can change the target category as you see fit, but you can also change the bordism category, go from d to $d+k$. You can also change the bordism category by adding structure to the cobordisms, like orientations or spin structures or tangential framings or stable framings and so on. All of those are sort of tangential structures and those are the ones I want to consider today.

I want to focus not on the details of gluing, which I'll sweep under the rug, or on the manifold invariants, but rather on the connection to higher structures.

There's been a rennaissance in classification relating higher structures to topological field theories in the last ten years. One type of classification result is the cobordism hypothesis, where you're looking at the fully local objects. We'll move to 3-dimensional theories and not fully local things in a few minutes, but first let's do some low dimensional example.

Here's an example of the kind of structure that emerges when you look at these.

Theorem 11.1. *(Lurie, Pstragowski) If you look at* $\mathrm{Bord}^{\mathrm{fr}}_{(0,1,2)}$*, this is free symmetric monoidal bicategory on a* 2-dualizable *object*

So what does 2-dualizable mean? There's a very general version, if you go from points to n-manifolds then it's on an n-dualizable object. In dimension 2 there's an easy way to state it. So first it is dualizable, so you have for x a dual ${}^*x$, and there's a evaluation and a coevaluation map

$$x \otimes {}^*x \xrightarrow{\mathrm{ev}} 1$$

$$1 \xrightarrow{\mathrm{coev}} {}^*x \otimes x$$

so that these compositions are the identity [pictures].

Then I have another condition, that the evaluation morphism has both a left and a right adjoint. These are morphisms in a 2-category so it makes sense to ask. So functors from this guy to any other target are the same as the 2-groupoid of 2-dualizable objects.

Let me do a more concrete example, if I look at

$$\mathrm{Fun}_{\otimes}(\mathrm{Bord}^{\mathrm{fr}}_{0,1,2}, \mathrm{Lin\,Cat})$$

this is the same as the 2-groupoid of finite semisimple categories. The correspondence takes the field theory Z to its evaluation on the point. This is the same as the category of Kapranov–[unintelligible]2-vector spaces.

There's also, these kind of semi-simple things appear, we're looking more at oriented theories, where we have a similar result, this is also, you could use Lurie's cobordism hypothesis or the presentation results in my dissertation, that, you can get an oriented version of the category, where you get a slightly different answer. There we get a functor $(\mathrm{Bord}^{\mathrm{or}}_{0,1,2})$ into linear categories, and that's equivalent to the same thing, finite semisimple categories plus the data of a non-degenerate trace which goes from Hochschild cohomology of the category to $\mathbb{C}$. This is sometimes called the *center* of the category, the [unintelligible]of the identity of $\mathbb{C}$. This is what you need to enhance to an oriented theory. Okay, so these are the sort of tools you have, the things you start seeing in the fully local case.

I don't want to talk about the fully local case, there are actually much older classification results. This is an older folklore theorem. If you don't go down to points, you can do $\mathrm{Bord}^{\mathrm{or}}_{(1,2)}$, the ordinary category, and this is the free symmetric monoidal category on a commutative Frobenius algebra object. It's a beautiful result, and it says that functors from this bordism category into Vect or wherever, this is the same thing as the groupoid of commutative Frobenius algebras in a classical sense.

Here this is implemented by taking your field theory and evaluating it on the circle. The real theorem behind this is saying that $\mathrm{Bord}^{\mathrm{or}}_{(1,2)}$ has a presentation with a generating object, the circle, and it has generating morphisms, and you only need four of them, cup, cap, pair of pants, and copants. [pictures]. Then you have relations, here, and also reflections of them, [pictures], and these are exactly the equations and relations that describe what it means to have a commutative Frobenius algebra. The upside down pair of pants gives a map $V \otimes V$ to V and this picture gives associativity. The next equation says that the cap is a unit. Then you get commutativity from this picture and here you get the Frobenius identity. This tells us that this geometric object is deeply connected to this familiar algebraic structure, this notion of commutative Frobenius algebra.

This persists.

I want to contemplate taking this and categorifying it, consider three-dimensional theories that are $1,2,3$-theories, valued in linear categories. What kind of structure will we start to see in these kinds of field theories. Just as we saw here in dimension 2, well, it starts in the same way, it has a generating object, generating morphisms like these, and then has generating 2-morphisms. So the relations will correspond to certain cobordisms, and you can turn a diffeomorphism into a cobordism by crossing a side with I and then using the diffeomorphism to parameterize the boundary. So we get generating 2-morphisms.

[missed some]

There are non-invertible 2-morphisms, such as these ones [picture] which express the fact that cap and cup are adjoint to one another.

So the theorem, which I'll write right here, and this is joint with Bartlett, Douglas, and Vicary, says that $\mathrm{Bord}^{\mathrm{or}}_{(1,2,3)}$ is the free symmetric monoidal bicategory on an (anomoly-free) modular tensor object.

[slides]

What we learn as a corollary of this presentation is that functors, symmetric monoidal functors from $Bord^{\mathrm{or}}_{(1,2,3)}$ into linear categories is the same as the 2-groupoid of anomaly-free modular tensor categories, a structure which has shown up in other talks at the conference.

[question about the framed case]

Okay, so, maybe, I don't know if I have enough time to talk about the fully local case, but it's interesting to compare, it's an interesting question when these extend to points. Since we're short on time I'll instead talk a little bit about how this is proven. Let me go back even to the two-dimensional case.

How do you prove that $\mathrm{Bord}^{\mathrm{or}}_{(1,2)}$ is free on a commutative Frobenius algebra object? The answer is to use Morse theory. Choose a Morse function on your cobordism, and it has some critical points, which give you a way, a prescription, for how to slice, and then in between the slices you have one critical point. Then you know you have handle attachments and then any cobordism can be written as a composite of these. To get the relations, you use a parameterized version of this. You instead look at a generic family of maps to the real line. Then Cerf theory says for any generic family, you'll have a family of Morse functions except at a finite number of times, when you'll either have a cubic singularity, a birth death point, or two critical points could exchange heights. In this family, you can trivialize the family, for any two bordisms they will be related by a path which can be cut into these piece where one such move is happening. Then there is diffeomorphism information where these join up with each other.

That's one way to prove this result, use Morse theory and its fundamental theorem and Cerf theory and *its* fundamental theorem. That's robust, that gives you a presentation in any dimension.

Morse theory was good because it let us decompose in a linear way. When we look at a two-category, we have a two-dimensional decomposition. The same technique can be applied except instead of mapping to the real line you map to the plane and get a singularity picture that is more complicated, so for example, you get [picture], instead of having isolated critical points, you get lines, cusp points, intersections, and that gives a two-dimensional way to isolate the places where something interesting is happening. Then you look at a parameterized version of this to get the relations. Looking at these elementary singularities tells you what the generators would be. By looking at the geometric version of that, you get [unintelligible]too.

With Douglas and Snyder we considered going from Bord_{0123} to the three-category where objects are tensor categories, morphisms are bimodule categories, 2-morphisms are functors, and 3-morphsims are transformations. We show that the three-dualizable objects are exactly the same as the fusion categories. We also show that if you have a spherical fusion category, then that gives you an oriented fully local three dimensional TFT. You can take that TFT and restrict it to just the 123 part. You can restrict it to this, it should give you a map from spherical fusion categories to anomaly free modular tensor categories, which is exactly the Drinfeld center. Then you can ask whether every 123 theory comes from a 0123 theory. We saw earlier the obstruction group, this is the so-called Witt group of (anomoly-free) modular tensor categories. You can see many examples of field theories can't be extended down to points.