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1. Alistair Hamilton Pretalk: Moduli Spaces, noncommutative geometry, and characteristic classes

I'll try to keep the material fairly elementary. The topics I hope to cover, I'll explain a theorem from Kontsevich about moduli spaces of rings. There's a theorem

Theorem 1.1. *(Kontsevich)*
The disjoint union of the moduli spaces of Riemann surfaces (with negative Euler characteristic), its homology, or rather the homology of the one point compactification, is the homology of a Lie algebra $\mathfrak{g}$. This is supposed to be a non-commutative analog of some vector fields.

I'll discuss this theorem, define the Chevalley-Eilenberg homology of a Lie algebra, talk about the Maurer-Cartan moduli space, and we'll use exponentiation to produce classes in the one point compactification of the moduli spaces.

I'll also discuss some super integral, what happens when you have an algebra of super functions, and then I'll discuss some elementary aspects of the BV formalism in finite dimensions.

To explain this theorem, I'll talk about the orbicell decomposition of this moduli space, due to Mumford, Penner, Thurston, Harer.

I'll talk about ribbon graphs. I can decompose this moduli space into orbicells which are ribbon graphs. I'll need the definiton of a metric ribbon graph, it's an ordinary graph with extra information, a cyclic ordering of the edges at each vertex and a positive real number attached to each edge.

[picture]

Given a ribbon graph like this, you can construct a surface with marked points. To each edge I associate a complex strip of width t_i and height $\pm\infty$. I glue these complex strips onto these edges. If I do that, I get something like this.

[picture]

At my vertices, I get some cyclic ordering. I glue these edges according to the cyclic ordering. I end up with something like this:

[picture] I get marked points at the infinities. At a vertex, I can define charts by looking at branches of $z^{2/n}$. I choose some branches in order to make some charts in the neighborhood, since each of the edges is $180°$.

At a marked point, in a neighborhood, I have a picture that looks like this. The horizontal lines become angles around a central marked point. Now I can label these points by the perimeter.

If I have two isomorphic metric ribbon graphs, they give me the same surface. So I have some map from $\mathbb{R}_+^{E(\Gamma)} \to M_{g,n} \times \Delta_{n-1}$, using the simplex to record the numbers. If I don't care whether my automorphisms preserve these numbers, then

I get a map modulo the automorphisms of Γ, giving the orbicells of the moduli space.

Let me now show that every Riemann surface is constructed from a graph.

Let me explain the Jenkins-Strebel theory in order to go back, using quadratic differentials.

This is a meromorphic section of the second tensor square of the complex cotangent bundle. A horizontal trajectory is a curve on the Riemann surface on which the quadratic differential is described by a positive function, pulling it back to this curve.

Likewise, we have the notion of a vertical trajectory, where the pullback is a negative function. To get a graph from the Riemann surface, we need a result that says, I have a Riemann surface with marked points labeled by positive real numbers, and now I invoke the result that says, there is a unique quadratic differential on the Riemann surface with the following properties:

(1) it has double poles at the marked points with residue $-\left(\frac{p_I}{2\pi}\right)^2$
(2) the closed horizontal trajectories are dense in the Riemann surface.

The horizontal trajectories form rings around the marked points, which carve out disks, and where they meet we'll get a graph. The vertices correspond to zeros of the quadratic differential. A vertex of valence $n+2$ is a zero of order n. The orientation gives us a cyclic ordering and the quadratic differential gives us a metric so we can decorate each edge with its length in this metric.

If I cut this on its vertical trajectories, these are rays coming out of these points, which will connect to the vertices. I'll end up if I cut on this with a bunch of complex strips. You can fill in the details, but you can see that the Riemann surface is obtained by gluing the strips as I said in the beginning of the lecture.

We have this theorem, Harer, Mumford, Penner, and Thurston,

Theorem 1.2. *If I take the disjoint union of $M_{g,n} \times \Delta^o_{n-1}$, the one point compactification i an orbicell complex.*

Let's see what happens near the boundary. As an edge length becomes zero, I contract that edge. The differential is $\partial\Gamma = \sum \Gamma/e$, where I don't contract loops. Why not? If I contract a loop on a Riemann surface, I'll pinch it. In this compactification, that's all my compactified point, so I'm sending this to zero.

Now I want to connect the homology of this thing to the homology of a Lie algebra.

The Lie algebra is $h[V]$ where V is a symplectic vector space. How is it defined? It's $T(V)/[,]$, so that's $\bigoplus V^{\otimes i}/C_i$, and I extend the inner product on V using the Leibniz rule, so

$$\{x_1, \ldots, x_m, y_1, \ldots, y_m\} = \sum \langle x_i, y_i \rangle x_{i+1} \cdots x_n \cdots x_{i-1} y_{j+1} \cdots y_m \cdots y_{j-1}$$

Then in this is $\mathfrak{g}[V]$ a Lie subalgebra, whose quadratic part corresponds to linear symplectic vector fields. $\mathfrak{g}[V]$ just starts the summation with $V^{\otimes 2}$ instead of $V^{\otimes 0}$.

Now the theorem is about the Chevalley-Eilenberg homology of this thing, so let me define that.

Let $(\ell, [\,,\,], d)$ be a differential graded Lie agebra with a bracket of degree one, so in particular it is symmetric. Take as my complex the symmetric algebra on it $(S^*(\ell), \partial)$ where the differential contracts with my Lie bracket:

$$\partial(\ell_1, \cdots, \ell_n) = \sum \pm [\ell_i, \ell_j] \ell_1 \cdots \hat{\ell}_i \cdots \hat{\ell}_j \cdots \ell_n ? \sum \pm d\ell_i \ell_1 \cdots \hat{\ell}_i \cdots \ell_n$$

I actually want relative Chevalley-Eilenberg homology, where I want to look at the coinvariants $S^*(\ell/k)_k$, with the differential induced.

Theorem 1.3. *(Kontsevich) $H_*(\sqcup M^{pt}_{g,n}) \cong H_(\mathfrak{g})$, but on the right I need to take direct limit and the relative Chevalley Eilenberg homology.*

So now I want a map from graphs to tensors in my Chevalley-Eilenberg complex. If I write my symplectic vector space in coordinates x_i and ξ_i, dual, then I put x_i on one end of and edge and x_i on the other end. For each one of the vertices, I'll get something in $\mathfrak{g}$, I put them in an order using the cyclic ordering.

I need to explain wwhy the differentials correspond. What is the differential for a ribbon graph? I contract the edges. But if I do that, I get a graph where I eliminate those two markings but concatenate the other markings, and bracket the dual variables.

Invariant theory tells me that this is an isomorphism.

[What about loops?] Well I only contract when I have two different tensors.

I need the definition of the Maurer Cartan moduli space. I take $\mathfrak{g}$, a pronilpotent graded Lie algebra. I form the Maurer-Cartan set, the set of elements of degree 1 so that $dx + \frac{1}{2}\{x, x\} = 0$. The part in degree zero $\mathfrak{g}_0$ acts on this thing:

$$exp(y)(x) = x + \sum_0^\infty \frac{1}{n+1)!}[ad\ y]^n(dy + [y, x])$$

Now $\frac{d}{dt}exp(ty)x = dy + ad\ y(exp(ty)x)$. So I'm just twisting by y a little bit. The quotient by this action is the Maurer Cartan moduli space $\widetilde{MC}(\mathfrak{g})$.

The Maurer Cartan moduli space are the cyclic A_∞ structures on [missed]. I don't have time to discuss that.

Now I'll talk about classes.

Take $MC(\mathfrak{g})$, there's a map ch to $C^*(\mathfrak{g})$, which takes x to $exp(x) = 1 + x + \frac{1}{2}x^2 + \cdots$, so really it's the completed Chevalley-Eilenberg complex. This is a cycle because $\partial(exp(x) = (dx + \frac{1}{2}[x, x])exp(x)$. We have an equivalence relation and Maurer-Cartan equivalent elements produce homologous classes. Suppose $x' = exp(y)x$. I can consider γ_y on $C^*(\mathfrak{g})$ which takes $g_1, \ldots g_k$ to $dyg_1 \ldots g_k + \sum[y, g_i]g_1 \ldots \hat{g}_i \ldots g_k$. This map is nullhomotopic, $id = (\gamma_y, s)$, with homotopy s which sends $g_1 \ldots g_k$ to $yg_1 \ldots g_k$. Why do equivalent elements produce homologous classes. Let the first one be $ch(exp(ty)x)$ and the second $exp(t\gamma_y)ch(x)$. These are both solutions to the initial value problem $\frac{dc}{dt} = \gamma_y(c(t))$ and $c(0) = ch(x)$. One of these is obvious and I did the other one earlier.

They both agree at 1, then, and so $ch(exp(y)(x)) = exp(\gamma_y)ch(x)$, and since this is nullhomotopic, this is homologous to $ch(x)$.

I'll say just a bit more about this in the main talk.

Let me say a few words about super integrals and the BV formalism.

Suppose I have σ a quadratic even super function on $\mathbb{R}^{n|2m}$. Say I take some polynomial superfunction $f(x, \xi)$ on this super space V. I want to make sense of integrals over V:

$$\int_V fe^{-\sigma}dxd\xi$$

I'll be picking out the term corresponding to the top dimensional thing. I don't have enough time to explain more fully than that. I can make sens of these integrals even if these things are not positive definite, using the Wick rotation.

Now I can describe the BV formalism in finite dimensions.

I have this symplectic vector space with an odd symplectic form. I define $p_\sigma[V]$ to be functions of the form $f(x,\xi)e^{-\sigma}$ with f a polynomial super function. A Lagrangian subspace is an isotropic subspace (the symplectic form vanishes) of maximum dimension (half the dimension of the original space).

The Batalin-Vilkovisky Laplacian is $\sum \partial_{x_i}\partial_{\xi_i}$, which has the property that $\Delta^2 = 0$. What is this machinery doing? We can see it as a reformulation of the normal exterior differential calculus. If I set $M = \mathbb{R}^n$, the "body" of V, then I can define a map $d : p[V] \to \Omega^*(M)$ which takes $f(x)\xi_{i_1}\cdots\xi_{i_k}$ to $f(x)dx_1\cdots \hat{dx}_{i_1}\cdots\hat{dx}_{i_k}\cdots dx_n$ I can check that $D\Delta = dD$.

Let me formulate the main theorems about this:

Theorem 1.4. $\int_L fe^{-\sigma}dxd\xi = \int_M D(fe^{-\sigma})$

In particular, integrating $-dxd\xi$ is a Δ-cocycle and two Lagrangian subspaces produce cohomologous cycles.

2. Alistair Hamilton: Main talk

In this talk, I'm going to describe some analogue of a theorem of Kontsevich, and as I explained, it allows us to recover the homology of the one point compactification of $\sqcup M_{g,n} \times \Delta_{n-1}$ (of negative Euler characteristic) as the homology of a certain Lie algebra, a noncommutative analogue of the Hamiltonian vector fields. I'll construct a differential graded Lie algebra whose Chevalley Eilenberg homology gives the homology of a certain compactification.

Kontsevich gave a construction to give classes in the moduli space. The initial data was a cyclic A_∞ algebra. In this talk, I'll use this theorem, the analogue, to produce some classes in a compactification. Here I should mention the recent work of Barannikov, who also describes a construction to give classes in the compactification. Our perspective will differ in that we'll use a different construction. This has some equivalence relation, so that equivalent elements give homologous classes. The classes we produce will live in a larger compactification.

We have this problem about constructing classes in the compactification. Extending classes can be described in terms of deformation theory.

It's quite familiar for anyone who knows about producing solutions to the quantum master equation in the BV formalism. I'll also describe a construction producing cohomology classes. The ideas go back to Kontsevich. Careful treatment was given in terms of modular operads by Chuang and Lazarev. The perspective I'll give today is by modifying a construction of Costello.

Now the question is, I have cohomology classes and homology classes, what happens when I pair the classes. I'll describe how this pairing can be described in terms of evaluating functional integrals over some finite dimensional space. This gives a way to test whether the classes we're producing are trivial or not. This perspective goes back to Kontsevich's original papers, but has never been carried out before. Today I hope I will have time to get to some examples where I check that what I get is non-trivial.

So my first job is to describe this analogue of Kontsevich's theorem. For this I need to talk about stable ribbon graphs, which will give an orbicell decomposition of the compactifications. There's a construction that takes a metric ribbon graph and produces a Riemann surface with marked points. The way that we go back to a metric ribbon graph is by using Jenkins-Strebel theory. This presents us with some problems if we want to apply this to nodal curves. If I have a nodal curve with marked points like this: [picture]

To use Jenkins-Strebel theory, I need marked points. I get a ribbon graph that looks something like this, I can't do something on this component with no marked points. Instead I just remember its topological type. These topological surfaces become the new vertices in my graph. Where the ribbon graph meets my surface I put boundary components, and then I put the half edges as the pairing of the boundary components. The graph that I get looks like this picture: The vertices are these topological surfaces.

Now like I said, since we can't recover the homolorphic structure of this component, we introduce a quotient of the Deligne-Mumford compactification, where we forget the holomorphic structure of the components with no marked points. This thing, I have $K\bar{M}_{g,n} \times \Delta^0_{n-1}$. The one point compactification of this is an orbicell complex.

There's a slightly larger compactification we can consider. We're recording the perimeters associated to these marked points. I can let these perimeters tend to 0. If I have a component all of whose perimeters are 0, I can't apply Jenkins-Strebel theory. I can get the complex structure if I have at least one positive value.

So this is the Looijenga compactification, I forget the complex structure when all perimeters are zero in a component. I'll denote this by $L_{g,n}$ and the Kontsevich one by $K_{g,n}$.

What's a differential in this orbicell complex? What happens when I let a length tend to 0? Let's look at the stable ribbon graph. It looks like this picture. What happens when I let the loop bounding a meridian of this torus to shrink? I get a surface that looks like this picture. The ribbon graph associated to it, I have a pinch point corresponding to different boundary components, and I get a loop between them. So contracting edges is given by replacing the edges with topological strips.

Now I have this compactification, an orbicell complex, and that's what the differential applied to an orbicell looks like. A sum over replacing edges with strips.

Now I'm ready to describe the analogue of Kontsevich's theorem.

How do I define the Lie algebra? Take a symplectic vector space V, define $h[V]$ to be

$$\sum_0^\infty V^{\otimes i}/C_i$$

modding out by cyclic permutations, and I extend the symplectic form to a Lie bracket using the Leibniz rule:

$$\{x_1 \cdots x_n, y_1 \cdots y_m\} = \sum_{i,j} \langle x_i, y_j \rangle (x_{i+1} \cdots x_{i-1})(y_{j+1} \cdots y_{j-1})$$

This is a Lie bialgebra. It has a cobracket $\nabla : h[V] \to h[V] \otimes h[V]$, I apply it to $x_1 \ldots x_n$ and it's the sum

$$\sum_{i,j} \langle x_i, x_j \rangle x_{i+1} \cdots x_{j-1} \otimes x_{j+1} \cdots x_{i-1}$$

Then ∇ extends to the Chevalley Eilenberg chains ℓ using the Leibniz rule. I multiply by a formal parameter γ, taking $k[\gamma] \otimes \ell$ and get $d = \gamma\partial + \nabla$. Then $d^2 = 0$ is equivalent to having a Lie bialgebra structure. Since ℓ is the symmetric algebra on h, this is $k[\gamma]\otimes S(k)\otimes S_{h\geq 1}$ so I can think of this as having two parameters. By specifying that the other parameter (ν) is zero I get another Lie algebra $\Lambda_\gamma[V]$. I get a diagram

$$\Lambda_{\gamma,\nu}[V]\to\Lambda_\gamma[V]\to h[V]$$

What's the analgoue of Kontsevich's theorem in this case? Glossing some technical, stuff, here's the diagram.

$$\begin{tikzcd}
H_*(\sqcup L_{g,n}) \arrow[r, equal] \arrow[d] & H_*(\Lambda_{\gamma,\nu}[V]) \arrow[d] \\
H_*(\sqcup K_{g,n}) \arrow[r, equal] \arrow[d] & H_*(\Lambda_\gamma[V]) \arrow[d] \\
H_*(\sqcup M_{g,n}\times\Delta^o_{n-1})^\infty \arrow[r, equal] & H_*(h[V])
\end{tikzcd}$$

I want to give an isomorphism. Let me tell you how to get from graphs to tensors. Suppose I have a graph like this. I want to give a tensor in the Chevalley Eilenberg complex. Each vertex will give me a tensor in the Chevalley Eilenberg complex.

[picture]

I get something like

$$(\gamma^{\text{genus}}\nu^{\text{boundary components}}(x_{i_1}\xi_{i_2}\cdots)(\cdots))(\cdots)$$

where I move over vertex surfaces and then their marked points.

Let me explain some applications. There's a way to produce classes in the Chevalley Eilenberg homology from elements in the Maurer-Cartan moduli space. I can use what's called the characteristic class construction. I have the same commutative diagram with those.

[Some discussion about the Maurer Cartan set]

I have the Maurer Cartan set, $x\in\mathfrak{g}[1]$ so that $dx+\frac{1}{2}[x,x]=0$, then I can exponentiate this to get $1+x+\frac{1}{2}x^2+\cdots$ Equivalent elements will give homologous cycles.

So suppose I want to lift a class from $\widetilde{MC}(h[V])$ in the homology of the one-point compactification that I want to lift to a class in the Looijenga space. One way is that I could try to lift a Maurer Cartan solution to $\Lambda_{\gamma,\nu}$. Let me filter this, so that $\gamma^g\nu^n h_1\dots h_k$ has degree $2g+n+k-1$. Kontsevich's thing sits there in filtration degree 0. Suppose I start with a Maurer-Cartan element h_0. If I want to extend it, I write it as $h_0+h_1+\cdots$, and I have to fill these other elements out. This filtration is compatible with the differential graded Lie structure in the sense that $[F_p,F_q]\subset F_{p+q}$ and $dF_p\subset F_{p+1}$. So if I compute $dh+\frac{1}{2}[h,h]$ up to order one, I get $dh_0+[h_0,h_1]+O(2)$. The first obstruction to extending this is that dh_0 (which is a cycle) should be a coboundary, using the differential $[h_0,\quad]$.

You can build an obstruction theory for this, and the cohomology is the cyclic Hochschild cohomology of my A_∞ structure. The Maurer-Cartan space of $h[V]$ are the cyclic A_∞ structures on V. I call these liftings quantum A_∞ structures.

The cohomology of the complex $\Lambda_{\gamma,\nu}$ with $[h_0,\quad]$.

Let's look at some examples. There is a family of cyclic A_∞ structures on a one dimensional space, due to Kontsevich. Look at the suspension of the groupnd field, then $h[\Sigma k]$, there's a family of structures $x=\sum a_i t^{2i+1}$ where I allow a_i to vary over k. It's a solution $\{x, x\} = 0$. The higher structurs are anything they like. The even ones vanish for degree reasons (sign reasons).

One thing that is known about these is that when you produce classes, these give the κ classes in the moduli space, claimed by Kontsevich and proven by Modello and Igusa.

Let's look at the cyclic Hochschild cohomology. This coincides with the cyclic Hochschild cohomology of the field in the usual case, generated by t^{2i+1}. It's easy to compute that, the differentials become zero. I start here and want to extend this to a quantum A_∞ structure. I need to apply $\nabla(x)$ to this thing. So $\nabla(t^{2i+1}$ is $(2i+1)\nu t^{2i-1}$. This is what you get. This thing here is a non-vanishing class in Hochschild cohomology. This is obstructed to extending. If I set $\nu = 0$, then this becomes zero. So it's obstructed to lift to $\Lambda_{\gamma,\nu}$ but in Λ_γ it extends trivially, without adding anything.

Let me formulate some results.

[Some discussion of grading]

If I take as data a contractible differential graded Frobenius algebra. I could take $\ll a, 1 \gg$ with $da = 1$ and $a^2 = 1$ Then

$$\int_{\mathbb{R}} e^{-(\sum a_i t^{2i+1} - \frac{1}{2}t^2)} / \int_{\mathbb{R}} e^{-\frac{1}{2}t^2}$$