GEOMETRY AND PHYSICS SEMINAR

GABRIEL C. DRUMMOND-COLE

1. I

'm going to start this talk from a very low base, and give you an idea even if you have very little idea about symplectic geometry. So to start off with, we'll be looking at Kähler manifolds (X, ω). So X is a complex manifold, its transition functions are holomorphic, and ω a compatible symplectic form, a closed non-degenerate two-form, with a certain condition for compatibility that I won't get into.

The invariants that we will mostly be concerned with are "holomorphic curves" $u : \Sigma \to X$, which are maps from a Riemann surface Σ, possibly with boundary, into X, which are holomorphic, respecting the complex structure on both sides.

We can study the moduli space of such maps, and such things always appear in finite dimensional families in symplectic manifolds. This means, in particular, the zero dimensional component is isolated maps, a collection of points, and we can count these to give us an invariant. These counts are called, very roughly, are called Gromov-Witten invariants.

Let's compute our first examples. For example, what is the number of degree one curves $u : \mathbb{CP}^1 \to \mathbb{CP}^n$ through two points. This number is one. This is, if you had not computed any until today, you just computed your first one. The number of degree 2 curves $u : \mathbb{CP}^1 \to \mathbb{CP}^2$ through five points is also 1. Now a more interesting one. The number of degree 1 curves on a cubic surface is 27. Now let's get even crazier. What about degree 1 curves on a quintic three-fold? This is $X = \{\sum z_j^5 = 0\} \subset \mathbb{CP}^4$. This is a smooth degree five hypersurface. The number is 2875. This has been known since the mid twentieth century, if I understand it correctly. But what about the number of degree two curves on X? There are 609250. The number of degree three curves is 317206375. There, our knowledge of Gromov-Witten invariants on the quintic three-fold stopped in 1990. Then physicists changed that, remarkably.

1.1. **Mirror Symmetry version 1.0.**

String theory, about which I know nothing, studies propagation of strings on Calabi-Yau Kähler manifolds (X, ω, Ω). Calabi-Yau-ness means you also have a holomorphic volume form, something that looks like $dz_1 \wedge \cdots \wedge dz_n$. The important feature as far as I'm concerned is that there is this nonzero section of $\Omega^{n,0}(X)$. The quintic three-fold is an example of this.

There are two models for closed-string invariants on X. This means strings that look like circles moving in X as I understand it. These are the A-model, the Gromov-Witten invariants of (X, ω), and the B-model, built out of the periods of Ω. Taking 3-cycles and integrating Ω over them.

We have two kinds of invariants: the Gromov-Witten invariants (A-model) and the periods of Ω, the B-model. The Gromov-Witten invariants depend on the symplectic structure and the B-model depends on the complex structure, the holomorphic volume form. This should look a little strange, saying that counting these should depend on the symplectic structure. In fact, the definition of Gromov-Witten invariants uses only an almost-complex structure. All you really need is compatibility with the symplectic form.

Physicists studied these and noticed that there are lots of pairs of manifolds $(X, \omega, \Omega) \leftrightsquigarrow (X^\vee, \omega^\vee, \Omega^\vee)$, so that these models of string theory are reversed. The A-model of (X, ω, Ω) and the B-model of $(X^\vee, \omega^\vee, \Omega^\vee)$ are in some sense equivalent, and vice versa. This was the observation. this is realy, this is the picture of what mirror symmetry is. There is some Kähler object, not necessarily a manifold, and a mirror object, and there are A and B things depending on the symplectic and complex structures with this kind of equivalence.

You recall, my story finished with degree three with Gromov-Witten invariants of the quintic three-fold. In 1991, string theorists had the ingeneous idea construct the mirrror $X^\vee$ to the quintic three-fold. I won't describe that mirror, it's easy to write down, there is a mirror. They computed the B-model on $X^\vee$, computed the volume form and integrated explicitly. The theory of periods can be assembled into generating functions satisfying Picard-Fuchs differential equations. They used this to predict the A-model of X would be same as the B-model of $X^\vee$. They predicted how this would continue. They predicted that it had a very interesting structure. This was extremely unexpected. Mathematicians set about proving it. In 1996, Givental proved this mathematically. Indeed the Gromov-Witten invariants have this structure and are equal to these numbers in all degrees.

This was a very interesting interaction of maths with physics. But mirror symmetry wasn't done. Oh no.

1.2. **Homological mirror symmetry.** Although in 1996 Givental proved this for all Calabi-Yau and Fano intersections in toric varieties, this wasn't good enough. In 1994, Kontsevich conjectured that this picture of mirror symmetry should be a reflection of a categorified version of mirror symmetry. The A-model of X was supposed to be the Fukaya category $\mathcal{F}(X)$, which I'll spend the rest of the talk explaining. The B-model should be the category of bounded derived category of coherent sheaves on X.

And so X and $X^\vee$ are mirror if $D^b\mathcal{F}(X) \cong D^bCoh(X^\vee)$ and vice versa. This should be an equivalence of triangulated categories. One question is why this is stronger than version 1.0. We can recover that version by taking Hochschild cohomology. $HH^*(\mathcal{F}(X))$ should be the Gromov-Witten invariants of X, and $HH^*(Coh(X))$ should be the previous B-model. Making this explicit is still, Kontsevich gave us a lot of work and is doing it. This is what he left us, that these categories encode the structure.

My plan for the rest is to explain what the Fukaya category is. Let me say a word about coherent sheaves. Since we're taking the bounded derived category, think of them as holomorphic vector bundles. The objects are $\mathcal{F}$ and the morphisms are $Ext*(\mathcal{F}, \mathcal{G})$, and composition is composition. Build a model from Cech cohomology, build this as a differential graded category.

1.3. **Fukaya category.** So, let's start with (X, ω) a symplectic manifold. Then the Fukaya category $\mathcal{F}(X)$ is an invariant which we associate to it. The objects of the category are Lagrangian submanifolds $L \subset X$. Lagrangian means $\omega|L = 0$ and

L is half-dimensional. For the morphisms, first define

$$CF^*(L_0, L_1) := \bigoplus_{x \in L_0 \cap L_1} \mathbb{C}x$$

Generically, these intersect in a finite number of points. These are supposed to span a vector space. We'll build the morphism space as the homology of some differential on this vector space. Define $\delta : CF^*(L_0, L_1) \to CF^*(L_0, L_1)$ by

$$\delta(x) := \sum_{y \in L_0 \cap L_1} \langle \delta(x), y \rangle y$$

where the coefficients $\langle \delta(x), y \rangle$ are the counts of holomorphic disks in X from X to Y bounded by L_0 and L_1, with corners mapped to x and y. Such families come in finite components. Then $Hom(L_0, L_1) := H^*(CF^*(L_0, L_1), \delta)$. Why does δ square to zero? That's the first question. The differential does square to zero and you can define composition, given x and y, you have $\langle x \circ y, z \rangle$ is the count of holomorphic triangles, and then the category has well-defined composition, and so on, and I should wrap up by saying that in my talk this afternoon, I'll give a proof of holomorphic mirror symmetry of the quintic three-fold.

2. Thomas Willwacher pretalk, deformation quantization

This is just a pretalk, so I'll talk about basic stuff. Even advanced graduate students won't have much to gain from the talk. The main talk later today will be situated in deformation quantization, what is that about? It was devised in the late 70s by several people as some mathematical formulation of what it means to quantize a physical system. When things are bigger than $\hbar$, things behave classically. But if we start with a classical system, can we always find a quantum system that behaves that way classically.

Let me say what I mean by a classical system. For me this is a smooth manifold M with a Poisson structure π. This gives you the Lie bracket on the space of smooth functions $\{\cdot, \cdot\}$. In local coordinates, if your manifold is $\mathbb{R}^{n+n}$ with the standard symplectic structure, with coordinates p and q, then

$$\{f, g\} = \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial g}{\partial p_j} \frac{\partial f}{\partial q^j}$$

and dynamically we should have a Hamiltonian $H \in C^\infty(M)$.

A quantum system is a Hilbert space $\mathcal{H}$ and a space of observables $L(\mathcal{H})$, linear operators on $\mathcal{H}$. What is quantization? It is a map from the algebra of observables to the algebra of observables that to lowest order reduces to the standard structure. That is, it is a map $Quant_\hbar : C^\infty(M) \to L(\mathcal{H})$. We don't have that this is a Lie algebra morphism to the operator bracket, but it should be modulo $\hbar^2$. So $Quant_\hbar(\{f, g\}) = [Quant_\hbar(f), Quant_\hbar(g) + O(\hbar^2)$.

The idea of deformation quantization is that most of the information is contained in the product structure on $L(\mathcal{H})$. So we don't need $\mathcal{H}$ itself. In principle, we can take the product and pull it back. The idea of deformation quantization is to put the star product on $C^\infty(M)$ related to this product on $L(\mathcal{H})$. So the idea is to study associative (not generally commutative) products on $C^\infty(M)$.

Definition 2.1. *A* star product $\bigstar$ *on M is an associative product on $C^\infty(M)[[\epsilon]]$. Morally you could think that ϵ is $\frac{\hbar}{i}$, but I don't want to get them confused. This formal power series is because analytic properties are hard. This should satisfy:*

- $\bigstar$ *is ϵ-bilinear*
- $f\bigstar g = fg+\epsilon m_1(f,g)+\epsilon^2 m_2(f,g)+\cdots$, *and m_i are bidifferential operators, depending only on the derivatives.*

There is one natural problem. If you started with a classical physical system. Given such a structure, can you always produce such a star product so that modulo $\hbar^2$, the star product is compatible with the Poisson structure. So, given π a Poisson structure, can we find a star product so that

$$[f,g]_\bigstar = \epsilon\{f,g\} + O(\epsilon^2)?$$

If the answer is yes, then classify all such star products.

Both questions were answered by Kontsevich in 1997. The answer is encoded in the so-called formality theorem.

If you have, I need some notation, but once you have it, you can answer both questions in a positive manner. You can always compute such a star product. Any star product arises in this form. You add these higher pieces $\epsilon\pi_1, \epsilon^2\pi_2$, and then plugging into Kontsevich's machine will generate everything.

There is one caveat, this looks like a physics problem, but physicists aren't interested these days in deformation quantization. It's possible but hard to do it in this framework. Even the hydrogen atom is hard. From a mathematical viewpoint, the answer is beautiful, the answer is maybe more interesting than the question.

So in general, in mathematics, how do we solve this kind of deformation problem? How do we study deformation problems? There is general machinery to do that. You give me some such structure, then I can produce for you a deformation context for this structure. This will be a differential graded Lie algebra $\mathfrak{g}$, with a differential, the bracket, and the differential is a derivation with respect to this bracket. The deformations you want, deformations of your algebraic structure, are in one to one correspondence with Maurer-Cartan elements in your Lie algebra, $\mu \in \mathfrak{g}^1$ so that $d\mu + \frac{1}{2}[\mu,\mu] = 0$. The ordinary Lie algebra $\mathfrak{g}^0$ acts on the Maurer-Cartan elements, as $x \cdot \mu = dx + [x,\mu]$. This gives you a natural equivalence, two guys are equivalent if they're in the same orbit. So we write $MC(\mathfrak{g})$/gauge.

For associative algebras, the correct deformation complex is the so-called Hochschild complex, $C^n(A) = Hom(A^{\otimes n}, A)$. On C^{*-1} there is a a Lie bracket, the Gerstenhaber bracket. Then for degree m and n elements, we can evaluate $[\psi,\phi](a_1,\ldots,a_{m+n-1}) = \psi(\phi(a_1,\ldots,a_n),\ldots,a_{n+m-1}) \pm \psi(a_1,\phi(a_2,\ldots,a_{n+1}),a_{n+2},\ldots,a_{m+n-1}) + \cdots$ until you are all the way at the end, and then you symmetrize that. This is in fact a Lie bracket. Before I give the differential, let me make one comment. If we have $m \in C^2(A)$, a map $A\otimes A \to A$, then this is an associative product if and only if $[m,m]=0$. In particular, when we have a product on A, we get such an element. It follows from this equation that $d_H = [m,\cdot]$ is a differential that is compatible with the Lie algebra structure. The Hochschild complex is this space with this differential and the Gerstenhaber bracket.

We should restrict to the maps that are differential operators in each slot, polydifferential operators. We consider $D_{poly} \subset C(A)$. How do we express the deformations we wanted to find? We have $\star = m_0 + \epsilon\cdots$, so we want to find things that start with ϵ. We want to look at the differential graded Lie algebra $\epsilon D_{poly}[[\epsilon]]$, and Maurer-Cartan elements here will be in one to one correspondence with $\bigstar$ products. Gauge equivalence gives a notion of equivalence on the right. We want to say that this is equal to a classical object, a similar object we can make from

Poisson structures. The relevant differential graded Lie algebra is polyvector fields $T^{\cdot}_{poly} = \Gamma(M, \wedge^{\cdot} TM)$, so for example $\pi \in T^2_{poly}$. Here you also have a Lie bracket, the Schouten bracket. Here for elements of degree m and n, we have $[\gamma, \nu]$, inserting ν into any slot of γ and subtracting the reverse, and this gives a well-defined product independent of coordinates. I'll leave the check to you, because we have to hurry a bit to get to the end.

In terms of this bracket, you can express the condition for π to be a Poisson structure. It is one if and only if $[\pi, \pi]$ vanishes. This is the same as saying for all f, g, h in $C^\infty(M)$, the Jacobi identity is satisfied for $\{\cdot, \cdot\}$. We can make a similar Lie algebra out of T_{poly}, and this will be $\epsilon T^{\cdot +1}_{poly}[[\epsilon]]$, and there is a natural gauge transformation here induced by the degree zero part, which will just be vector fields. The theorem by Kontsevich is that this set is isomorphic to the Maurer-Cartan elements of $\epsilon D_{poly}[[\epsilon]]$ up to gauge transformation. A star product comes from a Poisson structure, and every Poisson structure gives a star product.

Kontsveich proved that this statement follows from a more general statement, his formality theorem. Let us step back before I say it. We want to say that two Lie algebras have the same Maurer-Cartan set modulo equivalence. If these were isomorphic, that would follow. They are not isomorphic, as you can easily check. You might guess, it may be enough that they have the same cohomology. This turns out to be too weak. You could consider the cohomology is a Lie algebra, but this forgets something about the Lie bracket on the original algebra. You want to find a way to remember the parts it forgets. The right notion, an enlargement of Lie algebra that remembers the correct part, is an L_∞ algebra. The Kontsevich formality theorem states that:

Theorem 2.1. *(Kontsevich)*
There is an L_∞ quasi-isomorphism between $T_{poly} \to D_{poly}$.

Theorem 2.2. *If $\mathfrak{g}$ and $\mathfrak{h}$ are differential graded Lie algebras and there is an L_∞ quasi-isomorphism $\mathfrak{g} \to \mathfrak{h}$ then their Maurer-Cartan sets are isomorphic.*

We won't discuss this. This is in Kontsevich's paper. It's not that hard. I should give you a rough idea of what that means, that there's an L_∞ quasiisomorphism, and then I want to state how Kontsevich proved this theorem.

Let's say that $\mathfrak{g}$ is a dg Lie algebra. Then you can build the Chevalley complex of this algebra, this is $S^+(\mathfrak{g}[1])$, the symmetric tensor product on the shifted complex. The $+$ means that you don't include 1. You can use the Lie structure on $\mathfrak{g}$ to get a differential on this complex. This also has a Hopf [sic?] algebra structure, this is a differential graded coalgebra. Then an L_∞ morphism is a map of differential graded coalgebras $S^+(\mathfrak{g}[1]) \to S^+(\mathfrak{h}[1])$. So the target is cofree, and it is determined by maps $u_n : (\mathfrak{g}[1])^{\otimes n} \to \mathfrak{h}[1]$. This is a quasiisomorphism if u_1 is a quasiisomorphism of complexes.

Kontsevich wrote down an explicit formula, that had only one problem, it was local, $M = \mathbb{R}^n$. In this business. The local case is hard, but there is a general machinery to glue to get a global morphism. This goes under the name of formal geometry. The globalization step I will not give you. So this should give me a polydifferential operator, that eats functions and spits out another function. Each of the γ_n is a gadget with n slots that can act on functions to take derivatives. You make a graph where for each arrow in any polyvector field, you designate either a function or a polyvector field on which to act. So this will be a sum over Kontsevich graphs Γ.

$$u_n(\gamma_1,\dots,\gamma_n)(a_1,\dots,a_m)=\sum_\Gamma\left(\int_{C_{n,m}}\omega_\Gamma\right)D_\Gamma(\gamma_1,\dots,a_m)$$

The constant is the hard part, $C_{n,m}$ is configurations of n points in the upper half space and m points in the real line, modulo translation and scaling, compactified. The second thing is ω_Γ. This is $\wedge_{(i,j)}\frac{d\varphi(z_i,z_j)}{2\pi i}$, over edges. Use the hyperbolic geometry, and φ is the angle between the ray from z_i to ∞ and from z_i to z_j. This gives an explicit formula.

That is the end of my talk.

3. Thomas Willwacher, Deformation Quantization

Thanks for the invitation, I'm very glad to be here. What will this talk be about? Deformation quantization. This is interesting, maybe, because it's a meeting point for techniques coming from quantum field theory and homological algebra. Within this realm you try to understand how these two players interact. What I want to talk about today is the formality theorem of Kontsevich. There have been two proofs. One is using physics methods, which was found by Kontsevich, and the physics explanation has been given by Cattaneo and Felder. There is a proof from homological algebra coming from Dima Tamarkin. There is one more point here. Kontsevich constructed one formality morphism, Tamarkin gave infinitely many that depend on a choice of Drinfel'd associator. You can ask, for which parameter values are these the same, and ask how to change the parameters on the other side.

Before I start, I'd like to introduce some notation. Here is a recollection. The main players are polyvector fields and polydifferential operators. Polyvector fields are $\Gamma(M,\wedge^*TM)$, and for us, most of the time, M will be $\mathbb{R}^n$. This, called T_{poly}, is a Gerstenhaber algebra with the wedge product as commutative product and the Schouten bracket as Lie bracket. The other player is the polydifferential operators D_{poly}, special Hochschild cochains of $C^\infty(M)$, given by polydifferential operators. In each slot this is a polydifferential operator. There are also operations corresponding to the product and the bracket, the Gerstenhaber bracket, and the Hochschild differential is given by bracketing with multiplication. There is an analogous version of the product, but to be precise, only on cohomology, and this is the cup product. More generally, there is a brace algebra structure on D_{poly}, you can get operations $c\{c_1,\dots,c_n\}$, where you take c and insert c_1 up to c_n. You sum this with appropriate signs. In particular cases, you might take m for your c. If we have an algebra, these will vanish when there are more than two inputs, there aren't enough slots in m.

What is the formality theorem that we want to study? This says that these two objects are quasiisomorphic as L_∞ algebras. There exists such a quasiisomorphism $T_{poly}\to D_{poly}$. In particular the complexes are quasiisomorphic, which is the Hochschild Kostant Rosenberg theorem. The original proof by Kontsevich, he gave an explicit solution to this problem, writing formulas in the following form,

$$u_n(\gamma_1,\gamma_n)=\sum_\Gamma\left(\int_{C_{n,m}}\omega_\Gamma\right)D_\Gamma(\gamma_1,\dots,\gamma_n)$$

where the sum is over directed graphs with two kinds of vertices, type I and type II. There are precisely n type one vertices, and to each such graph you can associate an operator $D_\Gamma(\gamma_1, \dots, \gamma_n)$. You get ω_Γ by wedging one-forms over the edges (i, j), the one-forms $\frac{d\varphi(z_i,z_j)}{2\pi i}$, where φ is the hyperbolic angle between the geodesics from i to ∞ and from i to j.

This is only an L_∞ quasiisomorphism, it doesn't know anything about the wedge or the cup. I should add that Cattaneo and Felder have given an interpretation of these as expectation values in the Poisson σ-model.

Later, there was a conceptually very different proof by Dima Tamarkin. He realized that this problem is easier to prove if you solve a more difficult problem, finding a quasiisomorphism that preserves more structure, a G_∞ quasiisomorphism. You have a problem, because on the right hand side there is no obvious G_∞ structure. To put this structure on D_{poly} has been known as the Deligne conjecture. You want it to reduce to the standard structure on cohomology. You could always do this by transfer, but you want the L_∞ part to be the usual Lie structure. It turns out that this problem can be solved. There are several different solutions. These depend on a choice of a complicated algebraic object, a Drinfel'd associator. I don't want to decribe what these are, but I will give a conjectural answer to what this space looks like, which is that it is a torsor for the exponentional group of the free lie algebra on generators $\sigma_3, \sigma_5, \dots$. If we have solved this, we can state the problem of constructing a G_∞ morphism. The second step is done for us by homological algebra. Rigidity of T_{poly} says this. The G_∞ structure cannot be deformed, at least not in a GL_n-invariant way. So by the "usual" homotopy transfer, you can put some G_∞ structure on T_{poly}, and since T_{poly} is rigid, there is a map $T_{poly} \to T'_{poly}$, the weird structure.

There are natural questions you can ask.

(1) You have two L_∞ quasiisomorphisms, are they the same or not? Of course, Dima's morphism is parameterized by an infinite dimensional space, so you could ask, for which Drinfel'd associator? There are multiple solutions to the Deligne conjecture. The one for which I can give an answer is by using the formality of the little disks operad.
(2) Not unrelated, of course, can Kontsevich's morphism be extended to a G_∞ morphism, using similar combinatorics?
(3) Above you had an infinite dimensional choice of parameters. Different angle forms in the Kontsevich formalism wouldn't change things. How do you change the Kontsevich morphism?

The results that I am going to talk about are:

(1) Yes. I know only three associators explicitly, fortunately it's one of those, the AT associator.
(2) If you look at the picture $T_{poly} \to D_{poly}$, you have a natural action of the Gerstenhaber operad on the left and the brace operad on the right. It's simpler to construct an unnatural action of the Br_∞ operad on the left than the G_∞ operad on the right. I mean ΩBBr_∞. If we change the question in this way, the answer is yes. This is a bigger operad, you have many operations. The operations are labelled by operations in the bar construction of the Brace operad, which can be identified with configurations of points. So $u_o(\gamma_1, \dots, \gamma_n) = \sum \int_{c(o)} \omega_\Gamma D_\Gamma(\cdots)$. So we only integrate over a subchain given to us by o.
(3) I will talk about this a little bit later, but there are explicit actions by the ogroup that is essentially the same as the Grothiendieck-Teichmueller

> group. You have an explicit action with explicit formulas by the graph complex. The 0th cohomology contains this Grothiendieck-Teichmueller algebra, which acts on the set of Drinfeld associators. In particular, we know how to write this action explicitly.

Let me talk about how to show that, how to construct that.

What do we want? I will have to show you three things. I will focus on point number two, because then you can make a G_∞ morphism and show point one by using a result I showed two years ago with Paolo Severa about the formality of the little disks operad.

[Ezra: how does this relate to the almost-brace operator on vector fields?] [Answer too quick]

So on D_{poly} you have an action of Br, and you can make a colored operad *hom* Br_∞ that governs a brace algebra, a Br_∞ algebra, and a map of Br_∞ algebras from the second to the first. I keep tiptoeing around the word colored. The space of operation has m inputs of the first color and n inputs of the second color, and I consider for Br_∞ just those that have all their inputs in the second color,, likewise for Br and the firts, and the part that is internal is generated only by operations that have inputs in the second color. We have the configuration space of points in $\mathbb{R}^2$ up to translation, scaling, and compactification, and for *hom* Br_∞ the configurations of points in the upper half-plane, modulo translation, scaling, and compactification. You have a natural right module structure, inserting configurations, this is part of the swiss cheese operad, and you have a left action of the braces operad. Believe this, I won't have time.

The map $Br_\infty \to C(FM_2)$ was constructed by Kontsevich and Soibelman. If I showed you, you'd ask why there wasn't an explicit formula.

[Isn't this the Deligne conjecture?] I was hoping to get away with this. Any operad quasiisomorphic to the little disks operad acts on D_{poly}, yeah, I bent the terminology a little bit for the sake of show effect. In fact you can more or less copy the proof to get an almost natural map *hom* Br_∞ to $C(C_{n,0})$. This is all the top layer. Then there is the Feynman graphs stage. In the Br_∞ case it is the operad Gra, given by linear combinations of undirected graphs. There is an operad structure that I will ignore. This naturally acts on T_{poly}. There's an explicit formula. Why? You can write this as a C^∞ structure on $\mathbb{R}^{n|n}$. There is a symplectic structure. An edge is an action of the Poisson structure. You have an explic map $C(FM_2) \to Gra$. You can associate to a grop a differential form. Put a differential form on each edge. Between 1 and 2 will be $\frac{d\ Arg(z_1-z_2)}{2\pi}$. In particular the composition gives an action of Br_∞ on T_{poly}, almost explicit. On the *hom* Br_∞ part, you get Kontsevich graphs, $KGra$, graphs that have two kinds of vertices and are directed, and these form a bimodule over the two operads. Using Kontsevich's construction, you can get a differential form that acts on a chain. This basically completes the picture.

The construction is more or less explicit. There are no inversions. In a way, if you make the right definitions, the checks go smoothly. The main problem is that of signs. There are some tricks to handle those signs. I will maybe talk about one or two points of interest. You have configurations of points in the upper half-plane. Kevin asked, usually you would think that the Swiss Cheese operad plays a role. You'd normally consider configurations on the real line as well, the structure comes from the left action of braces. In particular, note the following. Say you have $\mathcal{P}$, an operad. You can make $\prod_n \mathcal{P}$ into a braces algebra. If you have $p\{p_1, \ldots, p_n\}$ then you sum up all ways of putting p_i into p. This endows the space with a brace structure. Similarly, if you consider the piece of the swiss cheese operad, configurations in $\mathbb{R}^2$, and then the swiss chees operad is an operad in modules over FM_2. So you get a right FM_2-module with an action of the braces operad. You should think of chains of $C_{n,0}$ as taking the sum over all $m \geq 0$ of putting m points on the line and letting them move arbitrarily. This gives a new chain in the total space of the Swiss cheese operad. By this construction, you can essentially get the braces action. This draws on the operadic composition in the Swiss cheese operad, implicitly encoded.

My time is up, this is a good place to stop.

4. David Ayala pretalk, November 10

I only have three pages, usually I have a lot more. I'm going to try to take my time. I'm going to talk about how to think of homology of a space via configurations with labels. I'll try to indicate how you can think of Poincaré duality along thes lines.

Fix A a commutative group and let X be a locally compact Hausdorff space. For this whole talk I'll be talking about the space of configurations $C_A^{nc}(X)$. I want nc to mean the words noncompact and I'll leave that out from now on. This is the space of finite unordered configurations of points in X with labels in A. I could say that this is a free A-algebra on X but as a definition, I'll say it's a space whose underlying set will be finite subsets Z of X along with maps $\ell : Z \to A$. As I've said, I didn't use that A is an algebra. I'll topologize this in a second, so that points can run together and when that happens I add the labels. I want it to be commutative so that when they run together things multiply. I'll describe a neighborhood of (Z, ℓ). Take a compact subset K and z in an open set V, then $U_{K,V}$ will be the pairs (z', ℓ') such that $z \cap K \subset V \cap K$ and, well, we can define a map $V \to A$, I'll call it $\bar{\ell}$, by assigning to any component the sum of the labels on points in that component. So that said, ℓ' on $Z' \cap K$ should make the diagram commute:

$$\begin{tikzcd} Z' \cap K \arrow[d] \arrow[dr, "\ell'"] & \\ V \cap K \arrow[r, "\bar{\ell}"'] & A \end{tikzcd}$$

When points run together, I add labels. I can view two points a and b as sitting inside V inside a large compact set, this should be in the neighborhood of a single point labelled by $a + b$.

Another feature of this topology is that points can disappear at ∞, that's the role of the compact set K. If K is the compact subset, another element, a configuration in the neighborhood described by this compact subset, I just erase all of these. Points that fall outside of any compact subset disappear. If X is $\mathbb{R}$, here's a loop. I can see a point come in from $-\infty$ to ∞ and at the basepoint the configuration is empty.

[Is this the symmetric product? This receives a map from SP^∞, but there you can't see points run off to ∞.]

Points can also disappear when they're labeled by the identity in A.

Let's see an example, with $X = \mathbb{R}$, and $A = \mathbb{Z}$. Then $C_{\mathbb{Z}}(\mathbb{R})$ has a map to the circle $(\mathbb{R} \cup \infty)$, where (z, ℓ) maps to $\sum \ell(z) z \in \mathbb{R} \subset S^1$. I claim that this is continuous. What happens when a point runs to ∞, it disappears, and that means it becomes the identity in $\mathbb{S}^1$, so it doesn't matter.

Fact 4.1. *Thi map $C_{\mathbb{Z}}(\mathbb{R}) \to S^1$ is an equivalence. [What about when A has a topology?] Well, you see $B\mathbb{Z}$ right here, but I'm not going to talk about the topological setting.*

There's a map $\mathbb{R} \times C_A(X) \to C_A(\mathbb{R} \times X)$ sending $(t, (z, \ell)) \mapsto (\{t\} \times z, \ell)$. If I have a bunch of dots on S^1 with labels, you just put this in the cylinder at coordinate t.

Here's the observation. This map factors through suspension, $\mathbb{R} \times C_A(X) \to \Sigma C_A(X) \to C_A(\mathbb{R} \times X)$. As $t \to \infty$, this circle goes outside any compact set and you have the empty configuration. There's an adjoint to this map, $C_A(X) \to \Omega C_A(\mathbb{R} \times X)$. I'll appeal to one fact, this is the same fact, more generally, that this is an equivalence.

For example, if we take X to be $\mathbb{R}^{n-1}$. Then $A = C_A(*)$ is eqiivalent to $\Omega^n C_A(\mathbb{R}^n)$. So $C_A(\mathbb{R}^n)$ is a $K(A, n)$.

Another example, if we take $X = \mathbb{R}^2$ and $A = \mathbb{Z}$, then there's an interpretation of, $C_{\mathbb{Z}}(\mathbb{R}^2)$, these are configurations in $\mathbb{R}^2$ labeled by integers, and I can build a rational function from that configuration, declaring the marked points as roots and poles, and you can see in this way that I won't get into that you get $\mathbb{CP}^\infty$.

We've seen that this is defined and seen examples. Let's see functoriality. A first functoriality property is, if we have a map $X \to Y$, do we get a map of configuration spaces? The answer is no, not generally. If we have a label running off the edge, it doesn't work. What if we say there is a map of one point compactifications $X^* \to Y^*$, then this induces a map $C_A(X) \to C_A(Y)$. This is where I want spaces that are locally compact. It sends (z, ℓ) to $(f(z), z') \mapsto \sum_{f^{-1}(z')} \ell(z)$. This sum will always be finite.

This is one functoriality property. Here's another. If we take $I \times X$, then $(I \times X)^* = I \times X^*$ [sic] and so we get a map $I \times C_A(X) \to C_A(I \times X) \to C_A(Y)$, so this does nice things with homotopies.

Another thing it does is $C_A(X \coprod Y) \cong C_A(X) \times C_A(Y)$.

More interesting, if $X \hookrightarrow Y$ is an open embedding, then there is an induced map in the other direction $C_A(Y) \to C_A(X)$, taking $(z, \ell) \mapsto (f^{-1} z, \ell|)$. The points may disappear, but that's allowed in this topology.

We get these restriction maps but even better, $C_A(\quad)$ form a sheaf, so that given an open cover $\{U_\alpha\}$ of X, then $C_A(X)$ is the equalizer of the diagram

$$\prod_{\alpha\beta} C_A(U_\alpha \cap U_\beta) \leftleftarrows \prod_{\alpha} C_A(U_\alpha)$$

The functoriality with respect to open embeddings gives us the following important observation. If $W \subset X$ is closed, then we get a restriction to the complement $C_A(X) \to C_A(X \backslash W)$, what goes to the basepoint? The fiber (not homotopy fiber) is $C_A(W)$. Here is the fact that is the hardest part of today, an application of Quillen's Theorem B, this is actually a homotopy fiber sequence. Putting all these facts about these noncompact configuration spaces together tells us that $\pi_* C_A(\quad)$ is a homology theory. Disjoint union goes to product, it's homotopy invariant, and this is the hard one, the requisite excision. This is not just any homology, but to know any homology theory, we test what it does to a point. This is A when $* = 0$

and 0 otherwise. This is singular homology. So $\pi_* C_A(\quad)$ is $\overline{H}(X^*, A)$, locally finite homology.

For some reason I had this written, that $C_A(\mathbb{R}^n)$ is a model for $K(A, n)$. I wrote that there for a good reason. This is an end of one half of the story. This is a non-compact version of Dold-Thom theory, where like what was mentioned they do $C_A(X)_c$, the compactly supported one, a subspace of $C_A(X)$, points can't go off to ∞, and you can topologize this as infinite symmetric products, and the hard work there is showing that an analogous sequence is a fiber sequence. They invented quasifibrations and went from there. We have two interesting things set up now. We know that the non-compact configuration space forms a sheaf. The compact one is not a sheaf. To know the global sections amounts to knowing it on local pieces.

So let's look at this on things made up out of $\mathbb{R}^n$ pieces, so n-manifolds. For simplicity I want a framing, so that these are locally canonically $\mathbb{R}^n$. These are not necessarily compact. Let's say that we're in as nice a situation as possible. We have a metric, a convex open cover. Let $\mathcal{U}$ be an open cover of M by copies of $\mathbb{R}^n$, agreeing with the framing, so that each pairwise intersection is also $\mathbb{R}^n$. This isn't required but it makes things easier. Then we see that $C_A(M)$ maps isomorphically to the limit over $\mathcal{U}$ of $C_A(U)$. Take $\mathcal{U}$ to be saturated, so it contains finite intersections. Up to homotopy, this is isomorphic to the limit of $C_A(\mathbb{R}^n)$, which are $K(A, n)$ spaces, so this is (maybe, I'll talk about it in a second) the limit of $K(A, n)$'s. The inclusions of the finite intersections into the larger pieces, all of the maps in this diagram are all equivalences. So this is (maybe) the space of maps from M into $K(A, n)$, not compactly supported maps or anything. The first maybe amounts to our hard fact from before. In general if you have a functor from a category into spaces sending all morphisms to equivalences, then it's easy to compute the limet of that diagram, it's the space of sections of the colimit of the diagram to a total space that you build in a canonical fashion. The fiber in the total space will be the value of the functor. The value of everything is $K(A, n)$, and this is a fibration with these fibers, and since this was a framed n-manifold, this is a trivial fibration, and the colimit is M. The space of sections of a trivial bundle is maps from the base (M) to the fiber.

You've just witnessed Poincaré duality. You've seen that the homotopy groups are homology, $\pi_* C_A(M)$ is $\overline{H}(M^*, A)$, and its' also $\pi_* Map(M, K(A, n))$, which is $H^{n-*}(M, A)$. When M is compact, one point compactification is an additional basepoint and then you get the regular Poincaré duality statements.

That's Poincaré duality using these noncompact configuration spaces. This sheaf property is key as far as I could tell. In what time remains, let me show that this functor from $eTop^{op} \to Top$ (the morphisms on the left are only open embeddings) remembers A, because you apply it to a point, you get the underlying set A, if you apply this to $\mathbb{R}$ you get $K(A, 1)$, and loops of that are A, a more direct way, if you look at compactly supported sections of the sheaf, $C_A(\mathbb{R})_c$, these are configurations of points on $\mathbb{R}$ with labels and if points are labeled by 0, they go away, but points can't run off to ∞. There's a map to A by adding everything together, this is an equivalence. This map didn't exist before, but it's here for compactly supported guys. Now given an open embedding of two Euclidean spaces into one, and given dots on either of them, in the compactly supported situation, I always get dots with the same labels, now I don't run into the problem when dots run off to the left or right. I get a map $C_A(\mathbb{R})_c \times C_A(\mathbb{R})_c \to C_A(\mathbb{R})_c$, which is a map $A \times A \to A$. This is an associative object in spaces, in the weak sense, which is the most we could have hoped for. If we play the same game in $\mathbb{R}^2$, you get, the picture, I get the multiplication of A again, which tells me that I'm recovering A as sort of a commutative object in a weak sense. You could replace 2 by a larger n and see more commutativity. I was moderately prepared to show you this hard fact. I'll indicate how it would go and then we'll call it a day. Let's have X sit inside $X \times (-\infty, \infty]$, a closed embedding, and the complement is $X \times \mathbb{R}$. The fiber sequence looked like $C_A(X) \to C_A(X \times (-\infty, \infty]) \to C_A(X \times \mathbb{R})$, and we wanted to see that it was a homotopy fiber sequence. Because of the $\mathbb{R}$ there. Well, I think I'm going to bail. You can think of the third space as a category, an algebra, and then the middle thing comes up as a Grothiendieck construction, a Borel construction, and that sets you up to use the Quillen theorem. If you wanted to hear more details I could go into it but maybe I'll stop there.

5. David Ayala, Geometry Physics Seminar, November 10, Weak n-categories of sheaves

I'm glad for this chance to be here, it seems like a well-informed audience. I'll be talking about joint work with Nick Roznblyum, a project to finding a resolution to there being so many situations where you have something like a category or n-category coming from geometry. I'm going to guide us there rather than take the shortest route. Let me do a really quick recap of Dold-Thom. They start with a based space $(X, *)$ and a commutative monoid A, and produce from this $Sp^\infty(X, A)$, which is a disjoint union over pairs $(X \times A)^k/\Sigma_k$, unordered configurations of points, up to some equivalence relations, $(x, 0) \sim \emptyset$, I can just forget those, $(*, a) \sim \emptyset$ and $(x, a), (x, b) \sim (x, a + b)$. The first of these I will leave as a variable, I can either have it or not.

Here are some highlights about the topology. Points can collide and you add their labels. Points labeled by 0 disappear, and if they run into the basepoint, they also can be forgotten.

The main theorem of Dold-Thom, or I don't know if it's the main one, say A is discrete and a group, then $\pi_* Sp^\infty(X, A)$ is the reduced homology $\overline{H}(X, A)$. You saw a variant of this if you went to the pretalk. This implies in particular that if you run this on $\mathbb{R}^n \cup \infty$, with basepoint ∞, then SP^∞ of this space is $K(A, n)$, which I may write $B^n A$.

Here is a question. What if A can only be delooped once? What if A is an associative monoid in topological spaces? We can't do the same thing because we've added things, so I don't know what order to add if A is not commutative. We definitely can't carry out this construction in general, but let's take $X = \mathbb{R}$, a linearly ordered topological space, and any finite subset inherits a linear order. In this way, any map $z \xrightarrow{\ell} A$, well, we know how to do multiplication, go to $\prod \ell(z)$. So $Conf(\mathbb{R}, A)$ is finite subsets Z of $\mathbb{R}$ and a map $Z \to A$. Salient features of the topology: points can collide and "add" labels, I know what order to collide in, and points with the identity label can disappear (or not, if there is no unit in A).

The real line wasn't the only space for which we can do this. If P is locally canonically linearly ordered space, we could do this. So let's take P, a large supply of such spaces are framed one-manifolds. Then we can form $Conf(P, A)$. I don't need to be able to multiply on the circle, I just need to know that just before they collide there's a linear order. This construction is nice and functorial in P in the sense that an embedding $P \hookrightarrow Q$ of framed one-manifolds induces a map of configuration spaces $Conf(P, A) \hookrightarrow Conf(Q, A)$.

So $Conf(\quad, A)$ is a functor from the category of framed 1-manifolds with embeddings to Top which takes disjoint union to product, and takes $\mathbb{R}$ to A by multiplying, and this map on $\mathbb{R}$ is an equivalence. In fact, we remember the structure of A as well by looking at the map $\mathbb{R} \coprod \mathbb{R} \to \mathbb{R}$, which induces a map of configuration spaces $Conf(\mathbb{R}, A) \times Conf(\mathbb{R}, A) \to Conf(\mathbb{R}, A)$. So this functor remembers (in a weak sense) the product on A. Your very own John Francis has a theorem that says all such functors that satisfy a certain condition (excisive) come from weakly associative monoids.

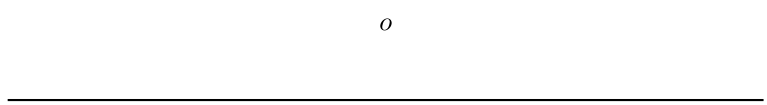
I think this is a very cool theorem because an associative monoid is something I think of as algebraic data, manifolds are geometric, and this interpolates. How far can this go? This was the kind of question that Nick and I, or I was interested in, I can't speak for him.

Think of a monoid A as a category with one object and the morphisms A, using the multiplication in A to define composition. Can you do this when you replace A by a category? It's not too hard to guess what you might do, $Conf(\mathbb{R}, \mathcal{C})$ consists of configurations where you mark points with morphisms and the gaps with objects. So this is finite subsets of $\mathbb{R}$ with labelling data. You can topologize this space in the way that you would guess, too, but you lose an important functoriality that you had when you were talking about an associative monoid.

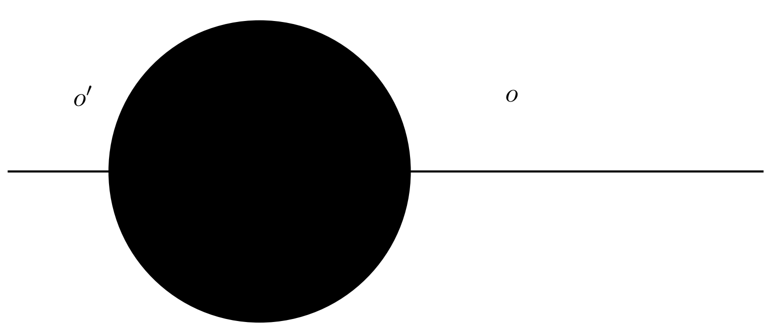
Can we have a theorem like John's that characterizes categories in terms of manifolds? Does $Conf(\quad, \bullet)$ characterize categories in terms of manifolds? This isn't functorial in embeddings of framed one-manifolds. An embedding e does not give a map of configuration spaces in any kind of good way. I can draw a picture for the problem. The tails of P are labeled by objects and they might not agree, if the ends of Q connect, well, we'd only get a map if the beginning source and end target are the same, but then you'd be talking about the underlying algebra. You can remedy this, but it does seem like you get a map going the other way, a restriction map. The picture, if I had points in Q, I can just restrict that labeling data. That's great. You get that map on underlying sets, but it's not continuous. Say a point runs off to the right, then in the restriction the point runs off to the end.

Retopologize the configuration space, and if you were in the pretalk, you have a pretty good idea of how you might do that. I'll call it $Conf^{nc}(P, \mathcal{C})$, for noncompact. I won't define this, but let me highlight some features. When points collide you compose morphisms. Points can disappear at ∞, that's the new part, and then the last part is that points with identity label can be forgotten. This is again optional, if your category is nonunital. If P is compact, it's the same as before.

I can draw a continuous loop

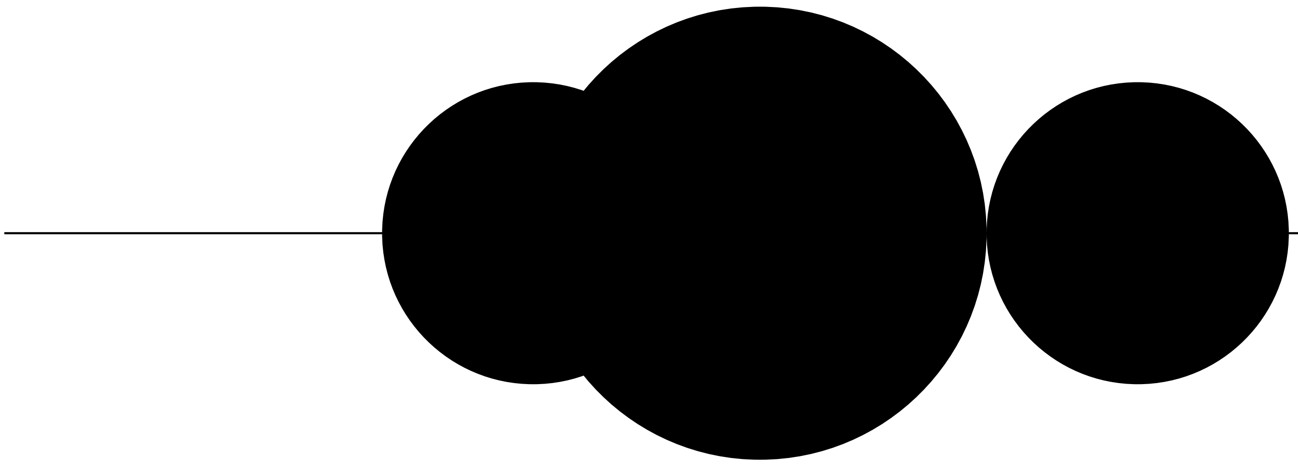


$t = 0$



$t = \frac{1}{3}$

So its enough to check it on $\mathbb{R}$, and $Conf^{nc}(\mathbb{R},\mathcal{C}) \cong |\mathcal{C}|$. If $\mathcal{C} = A$, then we get BA. It looks like we forgot information, we have only remembered the classifying space. The classifying space of the category is only up to homotopy equivalence. This particular point-set space remembers more than just the homotopy type. I can look $Conf^{nc}(\mathbb{R} \pitchfork \{0\})$. That's configurations that look like



You can make a deformation retract to forget the morphisms by pushing everything out to ∞. You can do similar pictures transverse to two points. These give morphisms and then restrict nicely to the other pieces. So you can build these into a simplicial space $\tilde{\mathcal{C}} : \Delta^{op} \to Top$ which takes $[p]$ to $\{(a_0 < \cdots < a_p), (Z, \ell) | Z \pitchfork a\}$. It turns out that $\tilde{\mathcal{C}}$ is a Segal space (weak category). I find units confusing. I haven't worked it out completely, but I don't see conceptual obstructions. Specializing to non-unital guys, completeness is not an issue. I only get monomorphisms, I don't get a full simplicial space. Those equivalences follow through to show that $\tilde{C}$ has a map to $\mathcal{C}$ regarded as a Segal space, which is an equivalence of weak categories. We keep this transversality, that's essential. I'll write a sequence of theorems.

Theorem 5.1. *(A.-Roznblyum)(n=1,nonunital)*
There is an adjunction between sheaves on framed 1*-manifolds with a notion of transversality and non-unital weak categories, which is an equivalence.*

Besides being a nice thing to know, this has a large supply of categories coming from geometry, and you have a sheaf on 1-manifolds, and you know when sheaves are transversal to submanifolds. Transversality means, I'll say this much, a consistent way to decide when $g \in \Psi(M)$ is transverse to a submanifold W. Going from non-unital weak categories is chiral or factorization homology. You get sheaves on all framed one-manifolds, can take sections, and then you get invariants. I would say we know most things we want to know about one-manifolds.

Theorem 5.2. *($n = \underline{n}$, non-unital)*
There is an adjunction and equivalence between non-unital θ_n-Segal spaces and sheaves on $Man_{\underline{n}}^{fr}$ with transversality

Take from this that we have on the left a model category, a model for weak n-categories, I learned it from Charles Rezk. This adjunction is an equivalence, that's the statement. I need to tell you what [unintelligible].

I'm running on the vague intuition about what an n-category is. I'm running on something precise. One feature of an n-category is that you have a hierarchy. So we would expect, in the geometry, to see a hierarchy. An object of $Man_{\underline{n}}^{fr}$ is submersions $M^n \to M^{n-1} \to \cdots \to M^0$ that looks like $\mathbb{R}^n \to \mathbb{R}^{n-1}$. An open cover is an open cover of each piece compatibly. I only need to know what it means to be transverse to points. It amounts to, a sequence of maps of finite sets $Z_n \to Z_{n-1} \to \cdots$, and I want to know when a sheaf is transverse to such subsets. For instance, if I take $\mathbb{R}^2$ projecting to $\mathbb{R}^1$, well, let me cut some things out and tell you theorem 80, we're 80 percent of the way

Theorem 5.3. *(n=n,nonunital)*
Non-unital weak n-categories with adjoints, there's an adjunction which is an equivalence of model categories between that and sheaves on framed n-manifolds with transversality.

This is a geometric setting familiar to geometers. This gives a way to construct weak n-categories with adjoints. A weak n-category has adjoints if for every k strictly between 0 and n, a k-morphism has an adjoint. In the two-category of categories, where $k = 1$, you know what it means. There's a lot that goes into how you might define this precisely. Going the other direction, a large class of examples of weak n-categories with adjoints are E_n algebras in topological spaces, and this is a categorical version of factorization homology.

Given a sheaf we get an n-category with adjoints. Given a weak n-category with adjoints, we get invariants by looking at sections of sheaves. This basically gives all of them.

Let me give some examples of sheaves that have a natural notion of transversality.

A sheaf is a functor satisfying a sheaf property. So there's the one that assigns to P^n all proper maps $W^n \to P^n$. The weak n-category you get is the bordism category $Bord_n$.

I decided to put this one in last minute, you could put in $W^d \subset P^n$, submanifolds, nothing has boundary, proper embeddings, I mean, and this gives an interesting n-category. If $d = 0$, I get the configurations we had before. If I say manifolds with some structure, then this gives E_n algebras when $d = 0$. I think that when $d = 2$ and $n = 4$ there is a relationship to Khovanov homology. It might remind you of knots in 3-manifolds and cobordisms of those. One last example is, it assigns to P^n the mapping space X^P or the space of, a stratification $C \subset P$, together with a map $P \to X$ that respects subcomplexes. There are lots of variants of this. You get the 3-category of conformal nets, speculatively, and it's probably time for me to stop. To finish, I gave three theorems. The stuff that Nick and I are working out, we're trying to do it as well as possible so that all of thesoe theorems happen at once. There's not much you need to input to get this flavor of theorem. It'll, the types of theorems that will come out are that sheaves on some kind of locally defined spaces with some kind of transversality is equivalent to some kind of algebraic data. These types of theorems. Say I consider topological spaces that are locally canonically $\mathbb{R}$ or $\mathbb{R}$ with a dot, then the structure you get on the other side is a nonunital category and a covariant functor.