VIRTUAL FUNDAMENTAL CHAINS AND APPLICATIONS TO LAGRANGIAN FLOER THEORY
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I want to talk about virtual fundamental chains and Floer homology. I've been working on this for twenty years but maybe I never made this kind of lecture on this topic. At first I was afraid it was too technical. I believe after three or four rewrites it's more usual. Five lectures are not enough, I think with a half-year course I could do it. But I can do maybe, give the statements and an idea of the proof. So the idea is as follows.

(1) Moduli space of holomorphic curves and its compactification.
(2) Kuranishi structures and orbifolds, which we'll need.
(3) (tomorrow) How to define the virtual fundamental chain. This is something like de Rham theory. These second and third lecture are similar to manifold theory. You define manifolds, differential forms, integration, and prove Stokes' theorem
(4) How to construct the Kuranishi structure on the moduli space of holomorphic curves
(5) Applications, some examples where Floer theory in this generality is useful in symplectic geometry.

Let me start with some basic things, about this moduli space. We have a symplectic manifold (X^{2n}, ω), a compact symplectic manifold. So ω is a 2-form and $d\omega = 0$ and ω^n never vanishes.

The thing considered by Gromov is this automorphism J of the tangent bundle of X, with $J^2 = -1$ and $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) \geq c\|v\|^2$. This is a "compatible almost complex structure" and then $g(v, w) = \omega(v, Jw)$ is a Riemannian metric.

We fix J and then there are many maps $\Sigma^2 \xrightarrow{u} X$ such that $JDu = Du\mathcal{J}$ where Σ is a Riemann surface.

On a non-integrable complex manifold, you usually can't expect any holomorphic maps from X to anything. So this is unusual, that there are many of these. So Gromov tried to use this to describe information about symplectic geometry. Various people recently think that this contains all the nontrivial global symplectic geometry, maps from Riemann surfaces.

You have flexibility and h-principle results and people thought at first that there were many things in between, and these are getting closer and closer. So people think that eventually we'll see that everything global comes from these holomorphic curves.

In a sense we expect, we know that this story is at least very rich. So, before going to the case, consider $g = 0, 1, 2, \dots$ and $\ell = 0, 1, 2, \dots$ and $\alpha \in H_2(X, \mathbb{Z})$ and we consider $\mathring{\mathcal{M}}_{g,\ell}(\alpha)$ which is the set of $(\Sigma, \vec{\mathfrak{z}}, u)$ satisfying some conditions up to an equivalence relation. So Σ will be a genus g Riemann surface, $\vec{\mathfrak{z}} = (\mathfrak{z}_1, \dots, \mathfrak{z}_\ell)$ are distinct points in Σ and u is a holomorphic map $\Sigma \to X$ (i.e., $Jdu = du\mathcal{J}$) with the homology class of α: $u_*([\Sigma]) = \alpha$. We say that $(\Sigma, \vec{\mathfrak{z}}, u)$ is equivalent to $(\Sigma', \vec{\mathfrak{z}}', u')$ if and only if there is a biholomorphic map $v : \Sigma \to \Sigma'$ such that $u' \circ v = u$ and $v(\mathfrak{z}_i) = \mathfrak{z}_i'$.

This is not compact and without a compactification this might be too wild. But there is a compactification which I'll explain in some cases later, which I'll call $\mathcal{M}_{g,\ell}(\alpha)$ which I'll call the moduli of stable map. The compactification is due to Kontsevich and can be used in almost all cases.

Then we have a map $\mathcal{M}_{g,\ell}(\alpha) \xrightarrow{(\mathrm{ev},\mathrm{fg})} X^\ell \times \mathcal{M}_{g,\ell}$, an evaluation and forgetful map. The evaluation map takes $(\Sigma, \vec{\mathfrak{z}}, u)$ to $(u(\mathfrak{z}_1), \dots, u(\mathfrak{z}_\ell))$. The forgetful map is more delicate. You take $(\Sigma, \vec{\mathfrak{z}}, u)$ to $(\Sigma, \vec{\mathfrak{z}})$ but then you have to shrink. I'll explain this process later on. This $\mathcal{M}_{g,\ell}$ is something called the moduli space of stable curves, introduced by Deligne and Mumford. If Σ is non-singular this is honestly $(\Sigma, \vec{\mathfrak{z}})$ but in the compactification you have to do some things.

The idea, I think from Gromov, is to take $(\mathrm{ev}, \mathrm{fg})_*([\mathcal{M}_{g,\ell}(\alpha)])$, and this is a class in $H_*(X^\ell \times \mathcal{M}_{g,\ell})$, a Gromov–Witten invariant, where the degree is $6g - 6 + 2\ell + 2c_1(X) \cap \alpha + 2n$ (or something like that, sorry). This space is singular, but people assume things about X so that $\mathcal{M}_{g,\ell}(\alpha)$ is good enough to have a fundamental class. Now they've done this algebraically and symplectically in some generality.

I want to talk about something a bit more difficult than this. This story I've told is more or less like a closed manifold. Even from the very early days people also studied the case when the Riemann surface had a boundary. In the case we've discussed, this is already difficult because it's singular, but at the end of the day you can define this class, which has applications. But now let's take $\Sigma = D^2$ a disk. Suppose you have a disk. You take a map from (D^2, ∂) to (X, L). The Fredholm theory needs good boundary conditions on the target, so L should be a Lagrangian submanifold and the real dimension of L is half the dimension of X and the symplectic form vanishes on X.

Maybe I should explain why this is good. Suppose we consider this moduli of u taking (D^2, ∂) to (X, L). This is a good moduli problem. But for other kinds of L, not Lagrangians, this is not good. For example, taking a very simple case, take $X = \mathbb{C}^2$ and $L = \mathbb{C}$, so this is not a Lagrangian (which would be $\mathbb{R}^2$). Then consider holomorphic disks $u : (D, \partial) \to (\mathbb{C}^2, \mathbb{C})$. But this means u_1 and u_2 are both holomorphic. Then u_1 has no boundary conditions and u_2 has zero at the boundary, so the first map is arbitrary and the second one is constant. Then the moduli space is infinite dimensional and this is not good. So $\mathbb{C}$ and $\mathbb{R}^2$ are both two dimensional but one is not very good.

If you naively do this with coisotropics, this will be too big, infinite dimensional. You need to do something clever. People have tried to do it but no one has succeeded. In the case of just half-dimensional with no isotropic condition, it's just a bad idea, on the other hand.

Weinstein is the person who promoted symplectic geometry for a long time, and he says that it's the study of symplectic geometry and its Lagrangian submanifolds. Something you don't see so much in Kähler geometry is Lagrangian submanifolds.

So I want to define a similar moduli space using this disk. So I'll define $\overset{\circ}{\mathcal{M}}_{k+1,\ell}(\beta)$ with $\beta \in H_2(X, L, \mathbb{Z})$. So this will be the set of $(D^2, \vec{z}, \vec{\mathfrak{z}}, u)$. So u is a holomorphic map from (D^2, ∂) to (X, L), so $JDu = Du\mathcal{J}$ (people write $\bar{\partial}u = 0$. Its relative homology class is β.

Then $\vec{z} = (z_0, \dots, z_k)$ are counterclockwise arranged distinct points on the boundary of the disk. Then $\vec{\mathfrak{z}}$ is again similar things, $(\mathfrak{z}_1, \dots, \mathfrak{z}_\ell)$ in the interior of the disk, distinct points, and we say that

$$(D^2, \vec{z}, \vec{\mathfrak{z}}, u) \sim (D^2, \vec{z}', \vec{\mathfrak{z}}', u')$$

if there is biholomorphic $v : (D^2, \partial) \to (D^2, \partial)$ such that $u' \circ v = u$ and $v(z_i) = z_i'$ and $v(\mathfrak{z}_i) = \mathfrak{z}_i'$.

So there's a compactification I'll explain soon, $\mathcal{M}_{k+1,\ell}(\beta)$ and you have an evaluation map $\mathcal{M}_{k+1,\ell}(\beta) \to L^{k+1} \times X^\ell$ using $u(z_i)$ for the first coordinates and $u(\mathfrak{z}_i)$ for the last coordinates.

You might try to take $\mathrm{ev}_*([\mathcal{M}_{k+1,\ell}(\beta)])$, this is the naive thing, in $H_*(L^{k+1} \times X^\ell, \mathbb{Q})$ but in the very *best* case, $Mm_{k+1,\ell}(\beta)$ is a manifold with boundary and you don't expect to have a fundamental class.

So for X and L a point and $\beta = 0$, this is $\mathcal{M}_{4,0}(0)$, this is the disk with four points given. It's a famous fact that you, you can change this by an automorphism of the disk so that z_0 is 1 and z_1 is $\sqrt{-1}$ and z_2 is -1. Then z_3 could be anywhere in the arc between -1 and 1.

Then $\overset{\circ}{M}_{4,0}(0)$ is an arc, and if you compactify it, it's a closed arc. So as such this naive idea does not work. If you try to get an invariant, try to take the fundamental class, it fails, and you have to do something. So the idea goes back to Floer, and it's to use these spaces $\mathcal{M}_{k+1,\ell}(\beta)$ to obtain structure on $H(X)$ and $H(L)$. You can't construct the homology element but you can use it to get some structure on this homology group. So then e.g., you get an A_∞ algebra on $H(X)$ if $H(L)$ vanishes. This is basically what the Floer homology is doing. It's a bit more complicated thing than usual Gromov–Witten theory. So in Gromov–Witten theory you get a kind of number. So you need to cook up a chain and prove some equality among chains in this case. So then transversality issues become bigger issues. I'll talk about that tomorrow but today I want to talk about this space in more detail.

This was an overview and now I want to start more serious work. I'll start by saying something about how to compactify—then later I'll talk about the fundamental *chain* of this space.

So first I want to talk about $\mathcal{M}_{k+1,\ell}$, this is the pure moduli space of genus zero curves with 1 boundary component with $k+1$ boundary marked points and ℓ interior marked points. So there are no handles, and there is only one boundary component, and we also have $k+1$ points on the boundary and ℓ in the interior, we only do not have X. So what is this $\mathcal{M}$?

Its elements are something like $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$, where Σ is a union of D^2s and S^2s, I want to do this a little informally. So you have a tree of disks [picture]—this is a tree-like union. It has no fundamental class, homotopy equivalent to a tree, and only double points. So it's forbidden to have three disks intersecting at one point. Then you add spheres [pictures], so trees of spheres attached to interior points of some disks. You are not allowed to attach these spheres at boundaries or double points. The spheres are also tree-like, so that loops in the configuration of trees are forbidden. Again you have only double points.

If you want rigorous definitions, you attach the usual Deligne–Mumford genus zero thing and then [unintelligible]. But it's easier to draw the picture.

Then z_i lies on the boundary of one of the D^2_a, these are the *irreducible components.* I require that $z_0, \dots, z_k$ respect the counterclockwise order. [Picture]

In a sense, if you take this as a topological space, you may think this has no orientation, so you put this in a plane preserving the complex structure of each component. The boundary marked points should respect this order.

The $\mathfrak{z}_i$ are in the S^2 or D^2, but in the D^2 its in the interior interior and it's not on a singular point, and all are distinct.

Now I want to say something about isomorphism and stability conditions. To be a *semi-stable* disk with marked points, I'll want to talk about that. So v will be an isomorphism between $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ if and only if it's a homeomorphism which is biholomorphic on each irreducible component, and then of course you expect that $v(z_i) = z_i'$ and $v(\mathfrak{z}_i) = \mathfrak{z}_i'$.

So we call $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ stable if its automorphisms are finite.

Maybe I can explain what is not stable. [Picture] This is not stable, this is in the case $k + 1 = 2$ and $\ell = 0$. This part, this is something like D^2 with one point, and all biholomorphic maps of D^2 which preserve this point, these are affine transformations of the upper half-plane, this is $z \mapsto az + b$, these are biholomorphic maps preserving ∞, so this is unstable. Another case is that if you have a component like this [picture]. This picture is in $\mathcal{M}_{4,2}$, and the components have no automorphisms except this one irreducible component is S^2 with 2 points. So $z \mapsto \alpha z$ preserves 0 and ∞. The conditions are then that disk components should have, let a be the boundary marked or singular points, and b be the interior marked or singular points, then this stability is that $a + 2b \geq 3$. In the case of S^2, this is, let a be the number of marked or singular poinds, then a should be at least 3.

Now we consider this $\mathcal{M}_{k+1,\ell}$, this is the set of $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ divided by isomorphisms, and this is a manifold with corners, and the dimension is $k + 1 + 2\ell - 3$. Then a representative $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is a codimension m corner if and only if Σ has $m + 1$ disk components.

For example, in the case of $\mathcal{M}_{4,0}$, you have your limiting configurations as like boundary of a family. But for interior nodes, you have a symmetry of rotations, so this is like parameterized by $[0, \infty) \times S^1$ and when you add ∞ it's a disk. So the first case it's a boundary point and in the second place it's an interior point.

In general, in higher genus, it will be an orbifold with corners. So $\mathcal{M}_{0,\ell}$, this has different issues. So $\mathcal{M}_{0,\ell}$ is not compact. The typical non-compact element looks like this [picture]. This is unstable, it has S^1, so finite automorphisms, this is unstable, but this is a limiting configuration. If you want to make it compact, you have to include it, it's not very bad because S^1 is compact. So this is why I write $k + 1$ because one boundary component makes it stable.

Next I want to include the holomorphic maps and then introduce the stable map homology. So I want to define $\mathcal{M}_{k+1,\ell}(\beta)$ where β is in $H_2(X, L, \mathbb{Z})$. So the points here are $(\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ where $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is *semi-stable* meaning they are tree-like configurations but I don't assume stability. So u is a continuous map $\Sigma \to X$ which takes the boundary to L which is holomorphic ($\bar{\partial} u = 0$) on each component and the relative class is β. For stability you need to consider the maps.

So $(\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ and $(\Sigma', \vec{z}', \vec{\mathfrak{z}}', u')$ are isomorphic via v if $v : \Sigma \to \Sigma'$ is a homeomorphism, biholomorphic on each component, preserving all the marked points and the map (as before).

The stability of this is as you expect, $(\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ is stable if the set of automorphisms v is finite.

Then the case, if you forget the map, it might be unstable, the configuration $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$, but with u it's stable. What's an example? Such an example is typically as follows. [picture]. If u is non-trivial on an unstable component. Then the map kills the automorphisms and makes it stable. The Riemann surface itself is unstable but after including the map it's stable. Many difficulties in handling this moduli space are related to this.

So I want to define a topology on $\mathcal{M}_{k+1,\ell}(\beta)$. The theorem says that it's compact and Hausdorff. Actually, Gromov never wrote down the precise topology, and the papers don't make it precise which topology they use. The difference in topology is big. If you want a function space, but the source is moving, you first need to identify, trivialize, the source. The source has non-compact automorphisms, so how to do that is a problem. This kind of topology is probably known to algebraic geometers. To define the topology first I want to define the forgetful map $\mathcal{M}_{k+1,\ell+1}(\beta) \to \mathcal{M}_{k+1,\ell}(\beta)$. This means we have $k+1$ boundary marked points and $\ell + 1$ interior marked points. So we have $(\Sigma, \vec{z}, (\mathfrak{z}_1, \dots, \mathfrak{z}_{\ell+1}), u)$ and you want to omit $\mathfrak{z}_{\ell+1}$. But the problem is that you might end up with an unstable thing.

That is, the issue is that $(\Sigma, \vec{z}, (\mathfrak{z}_1, \dots, \mathfrak{z}_\ell), u)$ might be unstable. Suppose you have this kind of thing [picture]. This is an element, and if you forget this marked point, then this component becomes unstable. If the map is nontrivial then it's okay but if the map is constant then it's unstable. So first we start with $(\Sigma, \vec{z}, \vec{\mathfrak{z}}_+, u)$ where $\vec{\mathfrak{z}}_+ = (\mathfrak{z}_1, \dots, \mathfrak{z}_{\ell+1})$ and you start by going to $(\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ where $\vec{\mathfrak{z}}$ is $(\mathfrak{z}_1, \dots, \mathfrak{z}_\ell)$, and then you shrink all unstable components.

I want to talk about universal families. So you have over $\mathcal{M}_{k+1,\ell}$ a universal family $\mathcal{C}_{k+1,\ell}$, where the fiber over $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is Σ. So for $\mathcal{M}_{g,\ell}$, in the Gromov–Witten case, the universal family is $\mathcal{M}_{g,\ell+1}$ and this is the forgetful map. This in fact turns out to be a universal family. So for $g = 1$ and $\ell = 2$, you get this picture [picture]. Then this point in $\mathcal{M}_{g,\ell+1}$, you have an element like this in $\mathcal{M}_{1,3}$, and if you forget this marked point this is unstable so you shrink it, so this element corresponds to this singular point.

You may think that this $\mathcal{M}_{k+1,\ell+1} \to \mathcal{M}_{k+1,\ell}$ is the universal family. This is almost true, but it's a bit different. This is a bit different so I want to mention it, so suppose you have something like this [picture]. You have a point $\mathcal{M}_{4,1}$ sitting above $\mathcal{M}_{4,0}$, and if you forget a marked point in the interior of a disk with two nodes, you collapse it. But this has moduli, a disk with one interior and two boundary marked points is parameterized by an arc. Then this whole arc corresponds to the singular point. So the fiber is not exactly Σ, it's Σ with boundary nodes replaced by arcs. So $C_{k+1,\ell}$ is obtained by shrinking arcs in $\mathcal{M}_{k+1,\ell+1}$. After shrinking, this is no longer a manifold but an orbifold. So outside the part corresponding to boundary nodes, this is a manifold.

This is about the universal family. You still get this and it's almost everywhere a manifold. Now once we remember, I want to define the topology, you should use neighborhood systems, but I'm going to use convergent sequences, this isn't quite right. First let me define $(\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ as *source stable* if $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is stable. If the source is stable then one can define the topology in the following way. Let $\xi_a = (\Sigma_a, \vec{z}_a, \vec{\mathfrak{z}}_a, u_a)$, and $\xi = (\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$. Assuming they are source stable, I'll define convergence $\lim_{a\to\infty} \xi_a = \xi_a$ (later I'll reduce to the source stable case). So first $(\Sigma_a, \vec{z}_a, \vec{z}z_a) = \eta_a$ should converge to $\eta = (\Sigma, \vec{z}, \vec{\mathfrak{z}})$ in $\mathcal{M}_{k+1,\ell}$. Then we want u_a to converge to u in some way. So now we can consider $S(\Sigma)$ to be the set of nodal points of Σ. Then $V(\eta)$ is a neighborhood of η in $\mathcal{M}_{k+1,\ell}$. You consider an ϵ-neighborhood of $S(\Sigma)$ in $C_{k+1,\ell}$, and consider $\Sigma \smallsetminus B_\epsilon(S(\Sigma))$.

We have this map φ from $\Sigma \smallsetminus B_\epsilon(S(\Sigma))$, so you consider this map

$$\varphi : (\Sigma \smallsetminus B_\epsilon(S(\Sigma))) \times V(\eta) \to C_{k+1,\ell} \xrightarrow{\pi} \mathcal{M}_{k+1,\ell}.$$

So this is an honest fiber bundle, if Σ has no singularity, this is honest. So you have a trivialization. If you have a singularity you have to remove this, and the image of φ contains $\pi^{-1}(V(\eta)) \smallsetminus B_\epsilon(S(\Sigma))$. Then the neighborhood of the singular point you can't parameterize nicely. Why is the universal family good?

So now $\xi_a \in V_\eta$, if you take $\pi^{-1}(\xi_a) = \Sigma_a$, and then we consider u_a restricted to this, So $x \in \Sigma \smallsetminus B_\epsilon(S)$, and you have two maps,

$$\Sigma \smallsetminus B_\epsilon(S(\Sigma)) \xrightarrow{\varphi(\ ,\xi_a)} \Sigma_a \xrightarrow{u_a} X$$

and $\Sigma \smallsetminus B_\epsilon(S(\Sigma)) \xrightarrow{u} X$. So the first assumption is that $x \mapsto u_a(\varphi(x, \xi_a))$ converges to $x \mapsto u(x)$ in the C^2 norm. [pictures]

We also need to control what happens near the node. The other condition is rather simple. We assume that for any δ there exists an ϵ so that U_a of any connected component of $B_\epsilon(S(\Sigma)) \cap \Sigma_a$ has a diameter smaller than δ. This part of the picture is the "neck region". If you have one component of that, forgetting that red part you want C^2 convergence. For that red part you request diameter small. You fix trivializations out of that part and on the neck part you request small diameter.

This is the definition of the topology in the case that everything is source stable. Now let me consider the general case, with $\xi_a = (\Sigma_a, \vec{z}_a, \vec{\mathfrak{z}}_a, u_a)$ and $\xi = (\Sigma, \vec{z}, \vec{\mathfrak{z}}, u)$ in $\mathcal{M}_{k+1,\ell}(\beta)$, and I want to define convergence without assuming source stability. I say they are convergent if and only if there is some m and $\hat{\xi}_a \in \mathcal{M}_{k+1,\ell+m}(\beta)$ and $\hat{\xi} \in \mathcal{M}_{k+1,\ell+m}(\beta)$ such that $\hat{\xi}_a$ and $\hat{\xi}$ are source stable satisfying two conditions

- the limit of $\hat{\xi}_a$ is $\hat{\xi}$, and
- the forgetful image of $\hat{\xi}_a$ is ξ_a and of $\hat{\xi}$ is ξ.

So you have to check some things to show that this is a topology. You need to know that these $\hat{\xi}_a$ and $\hat{\xi}$ exist.

Theorem 1.1. *The moduli space $\mathcal{M}_{k+1,\ell}(\beta)$ is compact Hausdorff (and in fact metrizable).*

This works in all genera. The compactness, Gromov compactness, that proof shows compactness, you need something but it's not very big. Hausdorffness you need something. What is important here is that that's where stability should enter. It's a quotient space. The quotient space fails to be Hausdorff when the group is noncompact. So stability is some restriction. The metrizability is another issue. You need to know that it's compact Hausdorff and satisfies the first axiom of countability. This is just an existence, the convergence definition.

The lemma is the following. Maybe I can just stop here, but here's an exercise

Lemma 1.1. *Let* $\lim \xi_a = \xi$ *and suppose* $\hat{\xi}$ *is source stable and forgets to* ξ*. Then there exists* $\hat{\xi}_a$ *source stable with limit* $\hat{\xi}$ *which forgets to* ξ_a*.*

I think you can prove it if you think a bit and then prove metrizability.

2. Lecture II

We introduced this moduli space in the morning, $\mathcal{M}_{k+1}(\beta)$, we'll need to include $\ell > 0$ for several things but let me consider $\ell = 0$ for simplicity. We have this evaluation map $\mathrm{ev} : \mathcal{M}_{k+1}(\beta) \to L^{k+1}$ which takes $(\Sigma, \vec{z}, u)$ to $(u(z_1), \ldots, u(z_k))$, and we want to define operations on $H^*(L)$. In a sense this, well, in some cases we have $L_1, \ldots, L_k$ with $L_i \pitchfork L_j$, and then we take

$$CF(L_i, L_j) = \bigoplus_{p \in L_i \cap L_j} \mathbb{Q}[p].$$

In our case, all L_i are L and so the transversality condition is never satisfied. So $CF(L_i, L_j)$ should be replaced with $H_*(L)$. If you have a Morse function, their critical points are discrete, and this is more like a Bott–Morse case. The critical point set is not discrete but it's a submanifold. To perturb L from itself breaks symmetry so it's better not to do it. To go directly to the homology group is sometimes not so nice, so we might try to obtain a chain model for the homology of L. For the virtual technique, you can probably use almost all chain models that you like. Which one you choose is related to what you are trying to do. Singular, Morse, de Rham, Cech, for each of these there is a version. Depending on the model you will have different problems. In this talk we'll use a de Rham model. In our book we use singular homology. This is sometimes difficult to do with intersection theory. If you teach singular homology it's technically heavier. The wedge product of de Rham forms is much easier than cup product.

So what do we do with the de Rham model? Instead of defining the homology group. we take m_k operations from $\Omega(L)^{\otimes k} \to \Omega(L)$, so we can write $m_K = \sum T^{\beta \cap \omega} m_{k,\beta}$, and these are some kind of A_∞-relations.

Once we cook these up on the chain model, then you can transfer the structure to the homology.

I want to mention the kind of one-line definition, take $h_0, \ldots, h_k$ on $\Omega(L)$, then I want to define $m_{k,\beta}$ via

$$\int_L m_{k,\beta}(h_1, \ldots, h_k) \wedge h_0 = \int_{\mathcal{M}_{k+1}(\beta)\, \mathrm{ev}^*(h_0 \times \cdots \times h_k)}$$

This is good only when $\mathcal{M}_{k+1}(\beta)$ is a manifold and $\mathrm{ev}_0 : \mathcal{M}_{k+1}(\beta) \to L$ is a submersion. In these good cases, you can use integration to define operations. But this rarely happens. There are many interesting cases when the map is not a submersion.

This is the main thing about the virtual fundamental chain technique. You want to adjust your integrations over a singular space.

The theory looks like manifold theory. You have (very singular) moduli spaces and want to justify integration over them. So I want to explain some framework for singular spaces to justify integration and prove, e.g., some kind of Stokes theorem. This kind of idea was studied by many people, but every group has their own taste and style. Closest to us is Dominic Joyce. But he seems to follow Grothendieck. He has about a thousand pages about his manifolds. He hasn't gotten to the homology theory yet. He's trying to define a kind of scheme or stack using smooth functions instead of polynomials. Hofer has an infinite dimensional version, this is very much in the vein of functional analysis. John Pardon, he wants to go directly to algebraic topology constructions to construct algebraic topology and these kinds of invariants. Probably he will not continue in this. If you like functional analysis you go to Hofer, if you like schemes you go to Joyce. If you like manifold theory you come to us, and if you like algebraic topology you go to Pardon.

So now I want to do the manifold theory type things. I want to explain a few things about orbifolds before going there. So $\mathcal{M}_{k+1,\ell}$ itself is a manifold, a cornered manifold, so you don't need to go to the orbifold world. But $\mathcal{M}_{k+1,\ell}(\beta)$ might have automorphisms. Now this could be nontrivial. Even in the genus zero case you need to go to the orbifold world not the manifold world.

Write $X = S^2$ and $L = S^1$. Now Σ is a disk with a sphere bubble. This has irreducible components Σ_D and Σ_S. So the disk goes to the upper half-sphere and Σ_d goes to the double cover of S^2. The automorphism of (Σ, z_0, u) is $\mathbb{Z}_2$ where it's the identity on the disk and $z \mapsto -z$ on the sphere.

I want to say a few words about orbifolds. I don't know, some people in symplectic topology think orbifolds are horrible and dangerous objects. For me, orbifolds are very simple. There are cases that are difficult, like maps between orbifolds. If you go to some, if you study non-effective orbifolds, you should be careful.

But since I don't want to make the story complicated or go to the difficult cases, I'll think of orbifolds as very close to manifolds.

I'll give a very elementary definition. Let X be a space. We consider a triple $U = (V, \Gamma, \phi)$, where V is open in $\mathbb{R}^n$ and Γ is a linear group action on $\mathbb{R}^n$ preserving V, and ϕ is a map from V to X, which is Γ-invariant (with respect to the trivial action on X) so that $\phi : V/\Gamma \to X$ is a homeomorphism to an open set. I assume effectivity: if $\gamma x = x$ for all x then γ is the identity. If a group acts trivially on a space, the local model is a manifold with a trivial γ action which might be twisted globally, so the orbifold structure is not unique.

So X is covered by U_a and the U_a are open sets with trivial Γ actions. You might think this was unique, but actually there are many different examples, because the transition functions when you glue them, this includes an automorphism of Γ. So there's something about the Cech cohomology of Γ. Then it's getting cumbersome to understand it. This is probably not so close to manifolds.

Now suppose you have $p \in V$ and you take Γ_p, the elements of Γ that fix p. Then you take $V_p, \Gamma_p, \phi|_{V_p}$, this is an orbifold chard, the "induced chart".

Now (I don't want to explain so much about orbifolds) if $v_i = (V_i, \Gamma_i, \phi_i)$, we say v_1 and v_2 are compatible if for p_i in V_i we have $\phi_1(p_i) = \phi_2(p_2)$. Then an induced chart at p_i is isomorphic to an induced chart at p_2.

This is the kind of definition that two orbifold charts are compatible. This is almost the same as what you get in manifolds, a little more complicated.

The orbifold structure of X is a cover $\cup \varphi_a(V_a)$, you cover X with locally finitely many charts that are compatible.

So what is a smooth structure? It's all compatible charts.

Then something I need is about boundaries. I said that to define morphisms of orbifolds is not trivial, especially for non-effective orbifolds. One example is the following. Let X be a point and Γ trivial. Then consider $Y = \mathbb{R}^2/\mathbb{Z}^2$ and if you want to find a map of topological spaces $X \to Y$, you send a point to the isomorphism class of 0. But this is a problematic map. One bad thing is, if we have a vector bundle then you can't pull back along this map.

The right thing is probably to think of orbifolds as a 2-category, if you try to do something involving equality of maps you may make a serious mistake. But anyway I'd like to restrict to a situation where I don't need to do this. So you have a map from orbifolds to topological spaces, and I'll say that two maps of orbifolds f and g are the same if their forgettings $\bar{f}$ and $\bar{g}$ are the same. For non-effective orbifolds, you don't get this property. Maybe someone used to 2-categories can do this without going to this kind of assumption.

So you can consider X and Y to be two orbifolds and want to know when a map f is an *embedding*. So first of all, it should be an embedding of topological spaces. Then you consider $p \in X$ and $q = f(p) \in Y$, and (V_p, Γ_p, ϕ_p) and (V_q, Γ_q, ϕ_q), and we can restrict to the case that $p = \phi_p(0)$ and likewise for q.

So I require $h : \Gamma_p \to \Gamma_q$ an isomorphism, and $V_p \hookrightarrow V_q$ is a Γ-equivariant smooth embedding compatible with f and ϕ_p and ϕ_q.

So locally it's an embedding and the isotropy group does not change. In the counterexample I gave the isotropy group changes.

So you can see that the composition of embeddings is an embedding. If you just take embeddings then the category of orbifolds is not so bad, it is just a 1-category.

Now I want to define vector bundles. If $\mathcal{E} \xrightarrow{\pi} X$ is a continuous map of orbifolds, this is a *vector bundle* if, for each p in X, you have a chart $(V_p \times E_p, \hat{\phi}_p, \Gamma_p)$ of p, with E_p a vector space and Γ_p acting on E_p linearly. Then you require compatibility of $\hat{\phi}_p$ and ϕ_p with projections.

I don't want to repeat but you can again define when two charts are compatible, and a global bundle is a cover by compatible bundle charts.

An example, take $S^2/\mathbb{Z}_p$, [picture], then you can consider the tangent space divided by $\mathbb{Z}_p$. The fiber is $\mathbb{R}^2/\mathbb{Z}_p$ at the endpoints so it's not the usual thing.

So the exercise is to suppose that, take a bundle, I don't want to say orbibundle, this is a vector bundle in the orbifold world in this sense. So $Y \hookrightarrow X$ is an embedding of orbifolds. Then we take $\mathcal{F}$ the fiber product of Y with $\mathcal{E}$ over X, and then you have a natural projection to Y and this has the canonical structure of a vector bundle. This is what I mean about pulling back vector bundles by embeddings. For example, the first bad map, where the point goes to something with isotropy, when you pull back you get something bad, you can't do that.

This is most of what I wanted to say about orbifolds. Now I want to define this notion of a Kuranishi structure. But first I want to define a Kuranishi chart.

Suppose that X is a topological space. Maybe we assume some other things, compact, Hausdorff, metrizable, I don't know. Then we'll take a Kuranishi chart of X to be $(U, \mathcal{E}, s, \psi)$ where U is an orbifold, $\mathcal{E} \xrightarrow{\pi} U$ is a vector bundle, s is an embedding $U \to \mathcal{E}$ and $\pi \circ s$ is the identity.

Finally, $s^{-1}(0) \to X$ is homemorphic to an open set.

So in a restricted setting, V a complex manifold, Kuranishi found a vector bundle $\mathcal{E} \to V$ and the map s. Then M_ϵ close to M is parametrized by $s^{-1}(0)$.

[I'm losing focus; he is not writing on the board very much, mainly just talking.]

Then the idea is the following thing. We have the moduli spaces, and you want to find, X is kind of singular and you want more structure, not just a topological space. Scheme is essentially something like this, locally write your space as a zero set of polynomials, and remember that. There's also a version in a complex analytic space. That case one can also handle by sheaf theory, but something bad in our case is that s is just a smooth function, so if you imitate schemes, you get things that you have to do on the sheaf of smooth functions (but this is a terrible thing).

So what do we do? We don't do sheaf theory. We focus on the zero set.

So a Kuranishi structure will be to have $(U, \mathcal{E}, s, \psi)$ as a chart, and define coordinate change of such an object. In place of taking only the zero set to have small neighborhoods and some consistency conditions. That's the notion I want to define, but it will take an hour, so how about a little break.

Okay so we have $\mathcal{E}$ over U and inside U w have $s^{-1}(0)$ and we map that to X, and that's a Kuranishi chart.

Now I want to define coordinate change between Kuranishi charts, supposing you have two of them. So a coordinate change $\Phi_{21} : U_1 \to U_2$, so you have an open set U_{21} inside U_1 and an embedding φ_{21} to U_2, and so you have $\mathcal{E}_1$ and sitting inside of it $\mathcal{E}_1|_{U_{21}}$ and this maps to $\mathcal{E}_2$ and this makes a square which should commute set-theoretically. So Φ_{21} should be $(U_{21}, \varphi_{21}, \hat{\varphi}_{21})$, and you assume that $\hat{\varphi}_{21} \circ s_1 = s_2 \circ \varphi_{21}$. The final condition is that $s_1^{-1}(0) \cap U_{21}$, this goes via φ_{21} to $s_2^{-1}(0)$, and then you can take ψ_2 to go to X. On the other hand you could have just gone by ψ_1. These should agree.

These are natural conditions, a pair of embeddings, these things should all be compatible, the Kuranishi map and the parameterization should commute. So the other things, the next one looks a little technical but you need it. So if $x \in s_1^{-1}(0) \cap U_{21}$ and $y = \varphi_{21}(x)$. Then you have a map from $T_y(U_2)/T_x U_{21} \to (E_2)_y/(E_1)_x$, we demand that this is an isomorphism, this is a normal bundle condition.

So $\Delta_X S_2$ will be in E_1 for $X \in T_x U_1$.

Why is this important? On U_1, X is approximated by $s_1^{-1}(0)$; on U_2 it's approximated by $s_2^{-1}(0)$. So this means that the two descriptions of X are consistent.

An example of this kind of embedding: so U_1 is the x-axis, and U_2 is $\mathbb{R}^2$. Then E_1 is 0 and E_2 is $\mathbb{R}$. Then s_1 is 0 and $s_2(x, y) = y$. So you increase the equations and the variables simultaneously.

So this is the notion of coordinate change.

Definition 2.1. A *Kuranishi structure* on X is, for each p in X, a tuple $\mathcal{U}_p = (U_p, \mathcal{E}_p, s_p, \psi_p)$, a Kuranishi chart,

(1) such that $\psi_p(s_p^{-1}(0))$ is an open neighborhood of p, so that,
(2) for $q \in \psi_p(s_p^{-1}(0))$, then there exists a (Kuranishi) coordinate change $\Phi_{pq} = (U_{pq}, \varphi_{pq}, \hat{\varphi}_{pq})$ from U_q to U_p so that $P_{pq} \ni o_q$ wher $s_q(o_q) = 0$ and $\psi(o_q) = q$.
(3) If q is in $\psi_p(s_p^{-1}(0))$ and $r \in \psi_q(s_q^{-1}(0))$, then $\Phi_{pq} \circ \Phi_{qr} = \Phi_{pr}$ from U_r to U_p.

In a sense, you can think the first two parts is a kind of definition of a manifold structure. In our case we'll need the third condition. In manifolds, the composition of smooth map and the inverse of the other, this is automatically smooth.

Definition 2.2. Given X and a Kuranishi structure $\hat{\mathcal{U}}$, a *differential form* on it is a collection $h_p \in \Omega(U_p)$ for all p such that $\varphi_{pq}^{-1} h_p = h_q$.

For twenty years we struggled to define morphisms. You have a pair of spaces and bundles, and you can try to write down some commutative diagrams. But suppose you have a manifold. Then we can say that $\hat{f} : (X, \hat{U}) \to M$ is a C^∞ map if you have $f_p : U_p \to M$ with $f_p \circ \varphi_{pq} = f_q$. We say $\hat{f}$ is *weakly submersive* if all f_p are submersions. Your map is a system of many maps, if all of them are submersions, then this is submersive. This is weak because U_p can have big dimension.

Before giving some examples, I want to say what we *want to* do and what we *can* do.

Theorem 2.1. *There is a Kuranishi structure on* $\mathcal{M}_{k+1,\ell}(\beta)$ *such that*

$$\mathcal{M}_{k+1,\ell}(\beta) \xrightarrow{\text{ev}} L^{k+1} \times X^{\ell}$$

is weakly submersive.

At the beginning this is a map of underlying topological spaces. But it lifts to be a submersion. Then the second theorem is the following.

Theorem 2.2. *If* $(X,\hat{\mathcal{U}}) \xrightarrow{\hat{f}} M$ *is weakly submersive then we can define a "CF-perturbation"* $\hat{G}$ *(I'll explain tomorrow), and you get* $f_!(\,;\hat{G}_{\mathcal{E}})$ *which takes differential forms on* $(X,\hat{\mathcal{U}})$ *to differential forms on* M *(for the choice* $\hat{S}_{\mathcal{E}}$*).*

Let me remind that if $f : N \to M$ is a submersion of smooth compact (or proper) manifolds, then you have $f_! : \Omega(N) \to \Omega(M)$, integration along the fibers,

$$\int_M f_! h \wedge g = \int_N h \wedge f^* g.$$

The theorem says that you can push out the differential forms in the presence of a weakly submersive thing, but this is more delicate.

So you can pull back differential forms very easily and this push out is much harder. So $\mathcal{M}_{k+1}(\beta)$ maps to L^k and to L. You can pull back and get $\text{ev}^*(h_1 \times \cdots \times h_n)$, but then we want to push out to get another differential form on L, the pushout is highly non-trivial. That theorem says that you have a weakly submersive map, then you can define a system of perturbations and get differential forms. If you make everything consistent at the boundary you get an A_∞ structure like you want.

This is a bit more complicated than manifold theory, where you can just integrate. So we don't get integration but you get perturbations to make sense of the integrations, and you can do this in a consistent way.

3. August 29: Lecture III

I'll talk today about something very similar to manifold theory. We learn first what is a coordinate and coordinate change, then what is map, differential form, and then integration of differential forms, and then the most interesting thing is maybe Stokes' theorem.

I want to start by reviewing Kuranishi structures. I want to draw a picture. You have $\mathcal{E}$ a vector bundle over an orbifold U with a section s and you have $s^{-1}(0)$ and from there $\psi_p : s^{-1}(0)$. For another point q you have $\mathcal{E}_q$ over U_q and then you have U_{pq} inside U_q which maps via φ_{pq} (covered by $\hat{\varphi}_{pq}$) to U_p, and everything must commute with ψ_p and ψ_q.

So this is a Kuranishi chart. Now $\hat{h} = (h_p)$ is a siystem of differential forms, this is a differential form on $(X,\hat{U})$, so $h_p \in \Omega^k(U_p)$ and $\varphi_{pq}^* h_p = h_q$.

Then k, the dimension of $(X,\hat{U})$ is the dimension of U_p minus the rank of $\mathcal{E}_p$. If s_p is transversal to zero, then this is exactly the dimension of $s_p^{-1}(0)$. But the issue is that this need not be transversal. But we assume that h_p is a differential form of this degree. So I want to integrate $\hat{h}$ over $(X, \hat{U})$. If this were transversal this should be something like

$$\int_{(X,\hat{U})} \hat{h} = \sum \int_{s_p^{-1}(0)} \chi_p h_p$$

where χ_p is something like a partition of unity. But suppose you have very singular maps, the zero sets will be singular, and there won't be a canonical way to define the integral in this non-transversal situation.

So I want to define an approximation regime, and the homological data you get will be independent but you'll need to remember the approximation data.

First I want to explain how we do it on one chart. Suppose we have just $\mathcal{U} = (U, \mathcal{E}, s, \psi)$, with $U = V/\Gamma$ and h is a differential form of compact support on a neighborhood of $s^{-1}(0)$. I want to define integrations on this zero set.

Definition 3.1. A *CF-perturbation* S on $\mathcal{U}$ is the data (W, ω, s_ϵ) where

- W is a real vector space, with an action of Γ
- ω is a differential form of top dimension of W (say d), which is of compact support, normalized to have integral 1 and non-negative.
- s_ϵ is a section of $\mathcal{E} \times W \to U \times W$ which is Γ-invariant and the limit as ϵ goes to zero converges to s in the C^1 sense.

We say S is transversal to 0 if s_ϵ is transversal to 0 on $U \times \operatorname{supp} \omega$.

I want to explain two things. This is the data abou perturbation we want to put. You are increasing the parameter by finitely many variables. Then it's easy to get the integration, I want to get

$$\int_{(U,S,\epsilon)} h = \frac{1}{\#\Gamma} \int_{s_\epsilon^{-1}(0) \subset V \times W} \omega \wedge h.$$

So h is a differential form on U, we can regard it on V, and so it induces one on $V \times W$ and ω is on W. This $s_\epsilon^{-1}(0)$ is a smooth submanifold of $V \times W$. Since we went up from U to V we divide by the order of the group.

Remark 3.1. If $s \pitchfork 0$ then we can take $s_\epsilon(x, u) = s(x)$ and then

$$\int_{s_\epsilon^{-1}(0)} \omega \wedge h = \int_W \omega \int_{s^{-1}(0)} h = \int_{s^{-1}(0)} h$$

so we don't have to perturb.

This is homologically well-defined. The local thing is not well-defined but the global thing is well-defined.

Some important message is that to make this locally you need some choice and typically, when ϵ goes to zero, you'd like this to give you the integral you want, but actually it diverges so you do it for positive ϵ.

Then the global thing, $(X, \hat{U})$, and $\partial(X, \hat{U}) = \varnothing$ and you have $\hat{h}$ a closed form (I'll be more precise later). Then if you take a consistent choice of S, then

$$\sum_p \int_{(X,\hat{U},S)\chi_p h_p}$$

is well-defined. This is like currents, and integration of currents is problematic, but if you have n-currents on an n-manifold and you do some approximations, you get a global thing even though the local invariants are well-defined.

Now when the boundary is non-empty (this empty boundary case is the Gromov–Witten case) you get operations but they depend on the choices and you can't get around this. But you want to make consistent choices over the different moduli spaces and get something at the end that is well-defined.

So for now I have these local CF-perturbations, or pointwise versions, I want a global version. I just explained you do this integration by partitions of unity. In manifold theory you have M and a differential form h. So you take

(1) locally finite cover $\cup U_\alpha = M$, with $U_\alpha \cong V_\alpha \subset \mathbb{R}^n$, then
(2) χ_a a partition of unity, and you do $\sum \int_{V_a} \chi_a h$.

So in the Kuranishi case we have uncountably many, and we want finitely many, a good coordinate system, since X is compact we can do it with finitely many.

In many situations people start from a good chart, not from a Kuranishi structure. For some people to associate U_p to all p you have too much structure, but you can't even do direct products with good charts. You can do it with Kuranishi structures.

I want to point out one very important point. In manifold theory, changing coordinates is symmetric, so you have go and back. But in the definition of the Kuranishi structure, φ_{pq} is not even locally a diffeomorphism. So you might not have a coordinate change in the other direction. That's this partially ordered set $\mathbb{P}$, and for each $\not{p}$ in $\mathbb{P}$, so I write $U_{\not{p}} = (U_{\not{p}}, \mathcal{E}_{\not{p}}, s_{\not{p}}, \psi_{\not{p}})$, the partially ordered set is finite, and

(1)
$$\bigcup_{\not{p}} \psi_{\not{p}}(s_{\not{p}}^{-1}(0)) = X$$
(2) if x is in the image of $\psi_{\not{p}}$ and $\psi_{\not{q}}$ then either $\not{p} < \not{q}$ or $\not{q} < \not{p}$
(3) if $\not{q} < \not{p}$ then there is a coordinate change $\Phi_{\not{p}\not{q}}$ from $\mathcal{U}_{\not{q}}$ to $\mathcal{U}_{\not{p}}$, and $\psi_{\not{q}}(U_{\not{p}\not{p}} \cap s_{\not{q}}^{-1}(0))$ is $\mathrm{im}\psi_{\not{q}} \cap \mathrm{im}\psi_{\not{p}}$
(4) On the disjoint union of $U_{\not{p}}$, the relation $x \sim y$ if $x = \varphi_{\not{p}\not{q}}(y)$ for $x \in U_{\not{p}}$ and $y \in U_{\not{p}\not{q}}$ is an equivalence relation.
(5) the disjoint union of $U_{\not{p}}$ modulo $\sim$ is Hausdorff

Theorem 3.1. *If you have a Kuranishi structure $(X, \hat{U})$ then there is a good coordinate system $\overset{\triangle}{U}$.*

We wrote various versions of this proof at different lengths. I think this proof is not something good to explain during talks. The six page version will be done soon. If you don't like it, then start from the good coordinate system.

So $U_{\not{p}}$ is an open set and if you glue together open sets of different dimensions this is not very good, the topology is pathological. So now we pass to $K = \{K_{\not{p}}\}$ where $K_{\not{p}}$ is compact in $U_{\not{p}}$ and

$$\bigcup_{\not{p}} \psi_{\not{p}}(\mathrm{Int}\, K_{\not{p}} \cap s_{\not{p}}^{-1}(0)) = X$$

still, and we call this a *support system* sometimes. Then $|K| = \coprod K_{\not{p}} / \sim$. We thought we should give a name for these spaces, and we call these heterodimensional compacta.

So let me explain [pictures]. So in this example, with X this circle in the plane with a complicated singular structure on the negative y side. If $p = (x, y) \in X$ for $y > 0$ then U_p is a neighborhood of p in X, and for $y \le 0$ it's a neighborhood of p

in $\mathbb{R}^2$. What does the coordinate change look like? If q is near p, but with positive y coordinate, then the arc is U_q. The one dimensional thing has an embedding to U_p. It's not, kind of, symmetric, you have a good map in one direction but not in the other. The number of equations may depend from point to point.

You try to construct these structures, heterodimensionality naturally appears. So how to construct a good chart? One is half of the circle, one dimensional, $U_{\not q}$, and the other is $U_{\not p}$, which is two dimensional. The compact things, these $K_{\not p}$ and $K_{\not q}$, and $|K|$ looks like this [pictures].

Something I want to define is the following thing, $\mathcal{CF}$ is going to be the sheaf of CF-perturbations on $|K|$. The important thing is that this is transversal to zero.

There are many ways to define sheaf theory. It's old-fashioned but I'll define the germ at each point and give the topology. At $p \in |K|$, I want to define $\mathcal{CF}_p$, this will be the set of $\not p$ in $\mathbb{P}$ such that $p \in K_{\not p}$. This is $\mathcal{P}(p)$, and an axiom is that there is some linear order. Now I write $\mathbb{P}(p)$ to be $\{\not p_1 < \cdots < \not p_m\}$ for some $m \in \mathbb{Z}_+$. So there we have $U_{\not p_1} \subset \cdots \subset U_{\not p_m}$, and on each of these you have the vector bundles $\mathcal{E}_{\not p_1} \subset \cdots \subset \mathcal{E}_{\not p_m}$, and everything has Γ actions. Then you consider (W, ω, s_ϵ), and W is again a finite dimensional vector space and ω is a form, a top form, with compact support and the integral of it is 1, and ω is a non-negative thing times the standard top form. Then s_ϵ is a section of $\mathcal{E}_{\not p_m}$, but if $x \in U_{\not p_k}$ then $s_\epsilon(x, w) \in \mathcal{E}_{\not p_k}$. Everything is Γ-equivariant and finally $s_\epsilon(-, w)$ converges to s in the C^1 sense as $\epsilon \to 0$.

The example is the following thing. [pictures] At this point p you have two charts, you have $\mathbb{R}$ in $\mathbb{R}^2$ with the zero vector bundle and the rank 1 vector bundle. Then you have these maps, and you take real valued functions that vanish on $\mathbb{R}$. This is a rather simple thing.

This S is transversal to zero if on the smallest set it's transversal to 0, if the restriction $s|_{U_{\not p^i}}$ is transversal to zero.

I believe it's kind of an exercise to prove that there is a global sheaf with germ like this. Before I say that I need an equivalence relation. I have this $(W^i, \omega^i, s^i_\epsilon)$ for $i = 1, 2$ and we say, well $S^1 \to S^2$ is a linear projection if we have a linear projection $\pi : W^2 \to W^1$ with $\pi_!(\omega^2) = \omega_1$. So $s^2_\epsilon(x, \pi(w)) = s^2_\epsilon(x, w)$. That's the projection.

Then if there is such a projection, then we say $S^1 \sim S^2$ and we take the equivalence relation generated by this relation.

So let me remind you the integration is something like

$$\int_{(s^1_\epsilon)^{-1}(0)} \omega^1 h,$$

which is the same as

$$\int_{(s^2_\epsilon)^{-1}(0)} \omega^2 h.$$

So this holds very elementarily. This gives the equivalence relation. So as I said, then of course this equivalence relation with respect to the filtration, the sequence of manifolds and vector bundles, I want to say that there is a sheaf $\mathcal{CF}_{\pitchfork 0}$ on $|K|$ such that the stalks at p are these equivalence classes over (W, ω, s). There is a standard thing in sheaf theory to make this a sheaf. So you can get this sheaf. Being transversal to zero means that $s \pitchfork 0$.

Maybe I want to explain two theorems.

Theorem 3.2. *This sheaf $\mathcal{CF}_{\pitchfork 0}$ is soft.*

Softness means for a closed subset $Z \subset |K|$, the global sections of the sheaf, the restriction from global sections to sections over Z is surjective.

So this is a section on its neighborhood, you can extend to a global thing without changing it on Z. This softness is a standard type of thing, a Thom type.

Before the break I want to state two or three theorems. Then after the break I want to say how to prove and use them.

Let me mention loosely, you have this chart, this Kuranishi chart, and you have something like a partition of unity associated to this, with support contained in this compact set $K_{\mathbb{P}}$. This is delicate because $U_{\not p}$ in $U_{\not p}/\sim$ is not closed, not open, so this makes partitions of unity delicate.

So cover $|X|$ by U_a and on U_a let S be represented by $(W_a, \omega_a, s_{\epsilon a})$, you can cover by a locally finite open set. Each chart has a representative. Then integration of this differential form with respect to this s_ϵ is

$$\int_{\overset{\triangle}{s}_\epsilon} h = \sum \int_{s_{\epsilon a}^{-1}(0) \cap (W_a \times U_a)} \chi_a h \omega_a.$$

So that's a very natural definition à la manifold theory. You want this to be well-defined, independent of the partition of unity blah blah blah. So in manifold theory this isn't totally easy but it's standard. Here this depends on ϵ and on the perturbation.

Finally Stokes' theorem:

$$\int_{(\overset{\triangle}{U}, \overset{\triangle}{S}_\epsilon)} dh = \int_{\partial(\overset{\triangle}{U}, \overset{\triangle}{S}_\epsilon)} h$$

and the proof looks like the usual one but you should make sure things are heterodimensional.

I want to explain how to prove and use this.

I want to prove this softness theorem, this is a transversal thing. Here W is not a vector space but a finite set, you can do the same thing. The existence in this case was in the 1996 paper, and that proof used a double induction. Then we expanded from 10 pages to 30 pages because of complaints. Using sheaf theory I can provide a shorter proof. The 30 page proof is too hard to explain in this kind of talk. The idea is using partitions of unity. So let me explain what I want to do. Let me just show existence of one global thing. So first $(\mathcal{CF}_{\pitchfork 0})_p$, I want to show this is nonempty. So I have $U_1 \subset \cdots \subset U_k$ and $\mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k$. So we have the section s, and I want to replace it with $U_1 \times E_1$ and take $W = E_1$ and ω any non-negative top form supported in a neighborhood of zero, and $\int_{\mathcal{E}}(x, w)$ is $s(x) + \epsilon w$, and this is trivially transversal. You want to extend this, you have U_2 and U_1 is contained in it with $\mathcal{E}_1$ in $\mathcal{E}_2$ over it. Now you consider U_2 projecting to U_1, and take $\mathrm{pr}^*(\mathcal{E}_1)$ and embed this in $\mathcal{E}_2$. That embedding is i. So now we have W_1 and ω_1 and $s_{\epsilon 1}$ and $s_\epsilon(x, w) = s(x) + i(s_\epsilon(\pi(x), w) - s(x))$. This is trying to extend to the neighborhood.

So the local construction is very easy. There is never a difficulty locally to making transversality.

So now I want to prove the global thing is non-empty. You take U_a that cover $|K|$ and you take $(W_a, \omega_a, s_\epsilon^a)$ which represent transversal sections on U_a.

So let me say what I mean by partition of unity, I say $f : |K| \to \mathbb{R}$ is smooth if $f|K_a$ is smooth for all a. An exercise is that for any open covering, there exists a χ_a, smooth, with sum 1.

So $W = \prod W_a$ and $\omega = \prod \omega_a$, and you write

$$\int_{[unintelligible]} (y, (W_a)) = \sum_a \chi_a(x) S^a_\epsilon(x, W_a).$$

But this is dangerous because it's like having the same W everywhere. So take x and consider a such that $x \in U_a$. Then $\Gamma_x \to \Gamma_a$ for all such a. This is a small orbifold with x in this neighborhood, so Γ_x is in Γ_a. So now take $\prod_{x \in U_a} W_a = W_x$ and ω_x is the same. Then we write that partition of unity:

$$\int_\epsilon^x (y, W_a) = \sum \chi_a(y) s_\epsilon(y, W_a)$$

[Some pictures]

So now we say that (W, ω, s_ϵ) is transversal if $\frac{\partial s_\epsilon}{\partial W}$ is surjective. That's enough. [couldn't follow]. So that's the existence of the transversal sections.

So now $\overset{\Delta}{S}$ is a global section and $h \in \Omega(\hat{\mathcal{U}})$ with top degree, and we want $\int_{\hat{s}_\epsilon} h$ in $\mathbb{R}$.

So we take $\bigcup U_a = |K|$ and $U_a \subset U_{\mathbb{P}(a)} \cap |K|$, and $U_{\not{p}}$ contains $K_{\not{p}}$. Maybe say $U_{\mathbb{P}(a)}$ is $\hat{U}_a$. You want to take a covering, so you look in the biggest thing containing it and take intersections. Now χ_a is a partition of unity with respect to this covering U_a.

Now χ_a is a C^∞ function on $\hat{U}_a$, and restrict it to give this partition of unity. So now $\hat{S}$ has a representative $(V_a, \omega_a, s^a_\epsilon)$, and I require that s^a_ϵ extends to $\hat{U}_a \times W_a$.

Then integration is similar:

$$\int_{hatS_\epsilon} = \sum_a \int_{\hat{s}_\epsilon^{-1}(0)} \chi_a \omega_a h_a$$

So you cover it like this [pictures] and most places you have a usual thing, but you have a strange situation in transitional places.

We need to prove it well-defined, but this is similar to the proof of Stokes'. There's one key statement. The dangerous part looks like where the boundary changes, the boundary of the compact set, the story may break some.

I want to explain one lemma. The key lemma we had to make it work is the following thing. Suppose we had $K_{\not{p}}$ in $U_{\not{p}}$ and we want $K^+_{\not{p}}$ also compact with $K_{\not{p}}$ in the interior of $K^+_{\not{p}}$ which is contained in $U_{\not{p}}$. So $|K_{\not{p}}| \subset |K^+_{\not{p}}|$ and put metrics on them and look at

$$\bigcup_{\not{p}} s^{-1}_{\epsilon_{\not{p}}}(0) \cap K^+_{\not{p}} \cap B_\epsilon(X),$$

and the lemma is that this is

$$\bigcup_{\not{p}} (s^{-1}_{\epsilon_{\not{p}}}(0) \cap \operatorname{Int} K_{\not{p}}) \cap B_\epsilon(X).$$

[pictures]

So this lemma actually shows Stokes', because the boundary of K_p gives you the extra terms and this shows that you don't have any such boundary.

That looks kind of strange, but let me try to prove it. There exists ϵ_0 such that for $\epsilon < \epsilon_0$ and

$$\bigcup_{\not{p}} (s^{-1}_{\epsilon,\not{p}}(0) \cap K^+_p) \cap B_\epsilon(X) = \bigcup_{\not{p}} (s^{-1}_{\epsilon,\not{p}}(0) \cap \operatorname{Int} K_p) \cap B_\epsilon(X).$$

My student Irie needed this for his work on loop spaces and he didn't like the proof, so I will explain his proof. So one direction is easy. Assume the lemma is wrong, we suppose that $\epsilon_m \to 0$, and fix $\not{p}$, and $x_m \in K_p^+$ and $s_{\epsilon_m,p}(x_m, W_m) = 0$ and $x_m \in B_{\epsilon_m}(X)$ but

$$X_n \notin \bigcup_q (s^{-1}_{\epsilon_n,q}(0) \cap \mathrm{Int}(K_q)) \cap B_{\epsilon_n}(X)$$

So $\lim_{n\to\infty} x_n = x \in X$. We know that X is contained in $\bigcup(s_q^{-1}(0) \cap \mathrm{Int}\, K_q)$. Take q so that x is in $s_q^{-1}(0) \cap \mathrm{Int}\, K_q$. So the condition we want to avoid is the following. [pictures]

So $TU_{\not{p}}|_{U_{\not{q}}}/TU_{\not{q}} \to E_{\not{p}}|_{U_{\not{q}}}/E_{\not{q}}$ is an isomorphism. Now $s_{\epsilon,p}|_{U_{\not{q}}}$ is contained in $E_{\not{q}}$ and also $s_{\epsilon,\not{p}}$ converges to s in the C^1 sense.

What this implies is that if you consider, $\vec{N}$ is the normal directions, $N_{U_{\not{q}}}U_{\not{p}}$, then $\nabla^{\vec{N}} s_\epsilon$ in $E_{\not{p}}/E_{\not{q}}$ is uniformly positive.

This normal derivative estimate says that $s_{\epsilon,\not{p}}(x_{\not{p}}, W)$ modulo $E_{\not{q}}$, this has norm larger than $cd(x, U_{\not{q}})$. This is C^1 convergence so still kind of non-zero, and the quotient is still zero and in the normal directions it's positive by something. If it's zero then the E_p component is zero so it should be right on this subset.

We write this as $s_\epsilon(x, w)$ is something like $(s_1^\epsilon(x), s_2^\epsilon(x))$ in E_p and E_p/E_q, and then

$$|s_2^\epsilon(x)| \geq cd(x, U_{\not{q}}).$$

Then after perturbation it's still zero on $U_{\not{q}}$ so this still holds so if the left is zero then the right is zero, so then if it's closed it should already be contained in $U_{\not{q}}$. It's not really difficult if you see what this normal bundle condition is a little carefully.

That's the end of the proof, so by this argument, you can show that even if the boundary of K looks exotic, everything looks like it's on the inside, you can do Stokes'.

4. August 30: Lecture IV

Today I'm supposed to talk about the existence result on the Kuranishi structure on the moduli space of disks. I need to define this structure on this space and explain how to use this integration to get A_∞ structures. The construction is in three preprints, with arXiv numbers 1710.01459, 1808.06100, and 1603.07026. The first one is a "gluing analysis" kind of heavy estimate papers. The two other ones assume the gluing analysis and are more topological. Estimates are difficult for me to do in front of the blackboard so I will focus on the second and third. I want to explain my goal and how to use it first.

Let me remind you that we have this moduli space $\mathcal{M}_{k+1}(\beta)$ with the class β in $H_2(X, L; \mathbb{Z})$, and this moduli space is of treelike unions of disks and spheres with $k+1$ marked points. This is a metric space, and the first theorem is that this has a Kuranishi structure.

First, to have individual Kuranishi structures is not good enough. To have just one is good enough for Gromov–Witten theory, because you have a closed manifold. Here you have a manifold with boundary and corners. So you should know how this looks at the boundary. So to say $\partial\mathcal{M}_{k+1}(\beta)$ doesn't really make sense, as this is just a metric space, but we can say naively that

$$\partial\mathcal{M}_{k+1}(\beta) = \bigcup_{k_1+k_2=k+1} \bigcup_{\beta_1+\beta_2=\beta} \bigcup_i \mathcal{M}_{k_1+1}(\beta_1)_{\mathrm{ev}_0} \times^L_{\mathrm{ev}_i} \mathcal{M}_{k_2+1}(\beta_2)$$

where $\mathrm{ev} : \mathcal{M}_{k+1}(\beta) \to L^{k+1}$. This kind of splitting is common in Floer theory.

Then the second theorem is that this equality is compatible with the Kuranishi structures. The decomposition has local charts and Kuranishi structures on them, and the second theorem is that the decomposition is compatible.

I want to explain the precise formulation, I think it's important to explain precisely what is to be proved rather than proving it.

So I want to define the *fiber product* of two Kuranishi structures.

So suppose $(X_i, \hat{U}_i)$ are Kuranishi structures and $\hat{f}_i : (X_i, \hat{U}_i) \to M$ is a map, with M a manifold.

Then $X_1 \times_M X_2$ is just the usual fiber product $(p,q) \in X_1 \times X_2$ such that $f_1(p) = f_2(q)$. In scheme theory to have fiber products of schemes has an important role. The virtual fundamental chain is some kind of intersection theory so this is central. But in manifolds, fiber products have problems. Categorically fiber products are well-defined and in schemes this is clean, but in manifolds this is not possible. In some point I have a dream to define a category containing manifolds faithfully and has honest fiber products. So then I need some assumptions.

I say that $\hat{f}_1 \pitchfork \hat{f}_2$. So I have $U^1_p \xrightarrow{f_p} M$ and $U^2_q \xrightarrow{f_q} M$. The transversality requirement is that $f_p \pitchfork f_q$. This means that the sum of the differentials of f_p and f_q at the origins gives the tangent bundle, $T_x M$, where $x = f_p(p) = f_q(q)$.

So you change the Kuranishi structure to make it bigger and then you can always get this kind of transversality, and this gives a possibility to solve the problem as described. If you can justify the category of Kuranishi structures, with embeddings, then you can maybe solve this problem. Joyce says you need to go to 2-categories.

Under the transversality assumption (which was weaker) you can always define fiber products. Then the claim is that $X_1 \times_M X_2$ has a canonical Kuranishi structure.

Then U_{pq} is the product $U_{1,p} \times_M U_{2,q}$. If M is an orbifold, then you have to be more careful. I don't know how much fiber products of orbifolds is understood. In principle it should be okay but you have to think a bit carefully how to do it. If you want to do the Floer theory for an orbifold you have to do it, I guess.

So now you can take $\pi_1^* \mathcal{E}_p \oplus \pi_2^* \mathcal{E}_q$ and then the section is $s_p + s_q$ and then on the zero section you can define $\psi_p \times \psi_q$.

The fiber product of good coordinate systems has some trouble. This fiber product is as described, and you need a coordinate change. Suppose p' is in the image of $\psi_{1,p}$ and q' is in the image of ψ_2, q. Then I have $U_{1,p'} \times_M U_{2,q'}$ and this maps to $U_{1,p} \times_M U_{2,q}$ via $\varphi_{pp'} times \varphi_{qq'}$. So suppose you have this good coordinate system and you have $\mathbf{p}_1$ with $U_{\mathbf{p}_1}$ and for $\mathbf{p}_2$ you have $U_{\mathbf{p}_2}$. You want direct products of good systems. Suppose $\mathbf{p}_1' < \mathbf{p}_1$ and likewise for $\mathbf{q}$. [pictures] For the Kuranishi structures you don't have these problems and then the fiber product is associative.

The categorical fiber product is automatically associative because it's unique. So we have to show associativity by hand. There is no room to do anything else, so there is no room for associativity.

So now I want to mention $\hat{\partial}$. Suppose given a manifold with corners M. If we have a manifold with corners, then what is the boundary? There is a notion of *normalized boundary* which maps $\hat{\partial} M \to \partial M$ (where the latter as a topological subset ∂M has clear meaning, but if you want ∂M to be a smooth manifold you have prolems). So $\hat{\partial} M$ is a manifold with corners and it's $k-1$ to one one the codimension k corners. [pictures]

So before this was a union but now it's a disjoint union, replacing ∂ with $\hat{\partial}$. So ten years ago we felt okay, but if you want to axiomatize it there might be other things. So the second thing, $\hat{\partial}\hat{\partial}\mathcal{M}$ should look like

$$\coprod(\hat{\partial}\mathcal{M}) \times \mathcal{M} \sqcup \coprod \mathcal{M} \times \hat{\partial}\mathcal{M}.$$

So this is something like

$$\coprod(\mathcal{M} \times \mathcal{M}) \times \mathcal{M} \sqcup \coprod \mathcal{M} \times (\mathcal{M} \times \mathcal{M}).$$

Each term comes up two different ways. You have these two things, and if you take the double normalizations of some N, then the corner points appear exactly twice. [pictures]

You have two canonical isomorphisms between the corners that have been blown up, one from associativity and the other from blowup. Coincidence of these two is not automatic—in general requiring just an existence theorem for an isomorphism is not enough, you need this kind of *corner compatibility.*

This is codimension 2 corners, but for higher codimension you have to do the same thing. Orbifolds is dangerous because they form a 2-category. You want to say that morphisms are isomorphic, not equal. I have some trouble saying that these are isomorphic, so that you need higher compatibilities. We'll work only with effective orbifolds and if things are set theoretically equal there, then they're equal.

Again, for the fiber product, you have set-theoretically well-defined fiber products so we just use those.

[digression on ∞-operads]

So the claim is the following. I defined the notion of CF-perturbations, I defined it only on a good coordinate system, I did it that way because you should go to a good chart. To do integration, you need a local finite cover, infinitely many charts are impossible to use. So you start with the good chart.

So now you can take fiber product. Let me remember what it is. Suppose you have these two charts, $(U_i, \mathcal{E}_i, \dots)$ for $i = 1, 2$ and you have $f_i : U_i \to M$.

So what's the perturbation? It's $(W, , \omega, s_\epsilon)$ where W is a vector space, ω is a top-form and s_ϵ is a section varying with ϵ. So you want a Fubini type thing so you want to take a fiber product of perturbations. So we assume f_1 is transversal to f_2. This is weakly transversal, and you define that f_1 is *strongly transversal* with respect to S_1 and S_2 ,you consider $(s^1_\epsilon)^{-1}(0)$ and $(s^2_\epsilon)^{-1}(0)$ and we want to require that $f_1 \circ \pi_1$ is transversal to $f_2 \circ \pi_2$ as a map on $(s^1_\epsilon)^{-1}(0), (s^2_\epsilon)^{-1}(0)$. Under this assumption one can take fiber products of these two data and still have this transversality. You need extra conditions so that you can take fiber products of CF-perturbations. Yesterday I proved that this sheaf is soft, and this sheaf of transversal perturbations is also soft.

Now we can take this fiber product, $S_1 \times_M S_2$. Suppose you have these strong enough things. For example, this is satisfied, you want to say that $f_1 \circ \pi_1$ restricted to $s^{-1}_{1\epsilon}(0) \to M$ is submersive, then it's transversal to anything else.

So suppose you have $\mathcal{M}_{k+1}(\beta)$ compatible with the boundary and corners. Then you can cook up $S_{k+1,\beta}$ transversal to zero and a CF-perturbation, and restricted to the boundary it's the disjoint union of the fiber products. That this conincides with the thing you already have is the condition.

So you write an order where $(k, \beta) \le (k', \beta')$ if and only if $\omega \cap \beta < \omega \cap \beta'$ or they are equal and $k < k'$.

Suppose we are given CF perturbations for smaller stages, then what we can do is the following thing, I want to emphasize a bit about corner compatibilities, note

$$\hat{\partial}\mathcal{M}_{k+1}(\beta) = \coprod \mathcal{M}_{k'_1+1}(\beta'_1) \times_L \mathcal{M}_{k'_2+1}(\beta'_2).$$

and the fiber product is given. You have your manifold with corners. [pictures]

[Discussion of the induction]

The nasty part of this story is the following thing. You have the Kuranishi structure and cook up good coordinates, and this induces something, a larger one, you go from $\hat{\mathcal{U}}_\partial$ to $\widehat{\mathcal{U}^+}_\partial$, and [pictures].

Theorem 4.1. *There are Kuranishi structures on $\mathcal{M}_{k+1}(\beta)$ compatible at boundary and corners.*

I don't want to prove this, it involves analysis that's too heavy. So let me instead define a notion called obstruction bundle data (OBD)

Theorem 4.2. *Given OBD on $\mathcal{M}_{k+1}(\beta)$ we get a Kuranishi structure on the same moduli space, canonical, in the sense of germs.*

The next thing,

Theorem 4.3. *There exists OBD.*

These two theorems are about individual moduli spaces. The next thing is, we define a notion of, some compatibility of OBD and I want to define it later on.

Theorem 4.4. *This compatibility of OBDs imply that $\partial\mathcal{M}_{k+1}(\beta) \cong \mathcal{M} \times \mathcal{M}$ with corners.*

The last thing is that there is a compatible system of OBD for these moduli spaces. This is probably more general. To construct OBD is geometric. You use the geometric intuition to construct this and then be done.

So I want to construct, mention some set. This is a bit dangerous set. $\mathcal{M}_{k+1}(\beta)$ will be a subset of $\mathcal{X}_{k+1}(\beta)$. This is the set of isomorphism classes of $(\Sigma, \vec{z}, u)$, with Σ a semi-stable disk, $\vec{z}$ its marked points (the same kind of object we used before to define moduli spaces) and u is a map from $(\Sigma, \partial\Sigma)$ to (X, L) which is not holomorphic but C^2, this is the only difference.

So to be in $\mathcal{M}_{k+1}(\beta)$ means that $\bar{\partial}u = 0$ and then we just replace with C^2.

I don't want to topologize this, because stability here is dangerous. If you're careful enough maybe you can do it but I'm afraid of it so I want to use it as a set.

Isomorphisms are the same, $(\Sigma, \vec{z}, u)$ is isomorphic to $(\Sigma', \vec{z}', u')$ if thre's v holomorphic, biholomorphic on components that intertwines u and u' and $\vec{z}$ with $\vec{z}'$. If you prove an equality it suffices to think in the set $\mathcal{X}_{k+1}$.

We'll need things outside of the moduli space for our Kuranishi charts. I want to define the notion of partial topology next. Suppose $\mathcal{M}$ is a metric space contained in a set $\mathcal{X}$. For each $p \in \mathcal{M}$, we specify the ϵ-neighborhood of p in $\mathcal{X}$. So we should have conditions

(1) $\bigcap_\epsilon B_\epsilon(p, \mathcal{X}) = \{p\}$
(2) for $q \in B_\epsilon(p, \mathcal{X}) \cap M$, there is a δ such that $B_\delta(q, \mathcal{X}) \subset B_\epsilon(p, \mathcal{X})$, and
(3) $B_\epsilon(p) \cap M$ for $\epsilon > 0$ is a basis of neighborhood systems for M.

So only elements of $\mathcal{M}$ have neighborhoods and they only have to have good behavior with regard to points in $\mathcal{M}$.

Lemma 4.1. *$\mathcal{M}_{k+1}(\beta) \subset \mathcal{X}_{k+1}(\beta)$ has a partial topology.*

The constriction says, take $\xi = (\Sigma, \vec{z}, u)$, and add $\vec{\mathfrak{z}}$to make it source stable, so that the triple $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is stable. So this is in $\mathcal{M}_{k+1,\ell}$ [pictures], this is ξ^+, and V is a neighborhood of ξ^+ in $\mathcal{M}_{k+1,\ell}$ and you consider the universal family $C_{k+1,\ell} \to \mathcal{M}_{k+1,\ell}$, and you remove an ϵ-neighborhood of the singular point set of Σ from the total space. Then you get this map φ_ϵ from $(\Sigma \smallsetminus B_\epsilon(S)) \times V \to C_{k+1,\ell}$ which commutes in $\mathcal{M}_{k+1,\ell}$ with the projection to V. This gives a local trivialization away from $B_\epsilon(S)$.

Then I want to define $B_\epsilon(\xi)$ so that you have this element $(\Sigma', \vec{z}', u')$ such that there exitss $\mathfrak{z}'$ so that $(\Sigma', \vec{z}', \vec{\mathfrak{z}}')$ is in V and the C^2 distance from $u' \circ \varphi_2([\mathit{unintelligible}])$ to u is less that ϵ. Moreover the u' of any connected component of $\Sigma' \smallsetminus B_\epsilon(S)$, the diameter is less than ϵ.

This is the same definition as for the stable map topology. Something that, this is an ϵ-neighborhood. If you just use this kind of definition, and if ξ is in the moduli space, it's okay, but if it's not in the moduli space you have problems. [picture]

We have to take some time to prove the axioms. It takes some time but it's something we need to do. Then we want to define,

Definition 4.1. *Obstruction bundle data*, say p is in $\mathcal{M}_{k+1}(\beta)$, represented by $(\Sigma_p, \vec{z}_p, u_p)$ and there is a neighborhood $B_\epsilon(p)$ in $\mathcal{X}_{k+1}(\beta)$. So for each x in here, you have $E_p(x) \subset \bigoplus C^2(\Sigma_{xa}, u_x^* TX \otimes \Lambda^{0,1})$ This is a target of the non-linear Cauchy equation, a finite dimensional subspace, Σ_x, of this set. For each p you fix a small neighborhood and for each point in the neighborhood you have this data.

1. $E_p(x)$ is semi-continuous with respect to p— for q in $B_{\epsilon'}(p) \cap M$, there exists δ such that $B_\delta(q) \subset B_\epsilon(q)$. The condition is that if $x \in B_q(\delta)$ then $E_q(x) \subset E_p(x)$. This finite dimensional subset depends on x and p. If you move p it gets bigger but never smaller.
2. $x \mapsto E_p(x)$ is smooth in x.
3. invariance of automorphisms.

This is delicate, this smoothness, regard u_x as a kind of $\mathcal{L}^2_k(\Sigma_{\text{thick}} X)$. So first of all, Σ_{thick} is the removal of a small neighborhood of a singular point. So I want to identify the domain of u_x with this thick part.

If you lose stability here then there is some ambiguity. Fix $\vec{\mathfrak{z}}$ so that $(\Sigma, \vec{z}, \vec{\mathfrak{z}})$ is stable, and assume that u is an immersion at $\mathfrak{z}_i$. [picture].

So start with u, pick your point $\mathfrak{z}_i$, and get a submanifold N_i of X with u transversal to N_i at $\mathfrak{z}_i$. Then there exists a unique $\vec{\mathfrak{z}}_x$ in any Σ_x such that $(\Sigma, \vec{\mathfrak{z}}, \vec{z})$ is C^2-close to $(\Sigma_x, \vec{\mathfrak{z}}_x, \vec{z})$ and $u'_x(\mathfrak{z}_i) \in N_i$.

Then you have φ_x which depends on everything which goes from Σ_{thick} to Σ_x. The issue is that this map canonically depends on Σ_x.

Now we consider $u_x \circ \varphi_x$ which goes in $C^2(\Sigma_{\text{thick}}, X)$. This is independent of x. Now I want to consider also, this, you have $E_p(x) \subset C^2(\Sigma_x, u_X^* TX \otimes \Lambda^{0,1})$, and the first requirement is that the support of s in $E_p(x)$ is contained in the image of φ_x. You can assume it.

Then we can use this φ_x, the C^2-image $C^2(\varphi_x(\Sigma_{\text{thick}}), u_x^*(TX \otimes \Lambda^{0,1}))$ mapping to $C^2(\Sigma_{\text{thick}}, u^* T_p \otimes \wedge^{0,1})$.

[pictures]

Then you have $C^2(\Sigma_{\text{thick}}, X) \to \Gamma_x(E_p(x))$ contained in $C^2(\Sigma_{\text{thick}}, u_p^* TX \otimes \Lambda^{0,1})$. Now you have x-independent spaces.

The claim is that for each k there is an ℓ such that when $e_1, \dots, e_m$ (m is the dimension of $E_p(x)$) which is a map $L^2_{k+\ell}(\Sigma_{\text{thick}} X) \to L^2_k(\Sigma_{\text{thick}}, u_p^* TX \otimes \Lambda^{0,1})$ such that e_i is a C^m map and

$$e_1(u'_x \circ \varphi_x), \dots e_m(u'_x \circ \varphi_x)$$

is a basis of $\Gamma_x(E_p(x))$.

So we use these fixed coordinates and parallel transport. The domain you can trivialize, and the original map is something like x-dependent function spaces. These are kind of hard to control, but anyway then the claim is that, you can just, as you take m to ∞, you need k and ℓ to be big. Then as far as this is $C^2_{k+\ell}$ you get this C^m-family. I think this is independent of the choices but we were lazy to prove this, so we assumed this was true for any such choice.

This is the definition of smoothness. Can I have five minutes to complete the story?

The first part is under these constructions to give the structures. You have $E_p(x)$ and for $p \in \mathcal{M}_{k+1}(\beta)$, this is just $x \in \mathcal{X}_{k+1}(\beta)$ such that $\bar\partial u_x \in E_p(x)$. I should make some transversality assumptions, that $E_p(X)$, at p, take $D_{u_p}\bar\partial$, this came from $W^2_{k+1}(\Sigma_p, \partial\Sigma_p), (u_p^* TX, u_p^* TL))$ mapping to $W^2_K(\Sigma_p, u_p^* T \otimes \Lambda^{0,1}))$. The assumption of this transversality is that the image of $D_u\bar\partial$ plus $E_p(x)$ is everything.

Now I assume this transversality and get this u_p. Something nontrivial written in our third paper is that this is an orbifold. You need this obstruction bundle, you need to solve these equations $\bar\partial u_x \in E_p(x)$, you can use a function theorem at ∞ if this moves nicely enough with x. Something delicate is when x has a node. [pictures]

Roughly you use some trivializations to do some functional analysis. The obstruction bundle is actually very simple, $(\mathcal{E}_p)_x = E_p(x)$ and $s_p(x) = \bar\partial U_x \in E_p(X)$, this is smooth by the conditions. Then $s_p^{-1}(0)$, then $\Sigma_x, \bar z_x, u_x$ is actually in $\mathcal{M}_{k+1}(\beta)$. So the Kuranishi neighborhoods are now automatic.

Let me finally say some words about coordinate change, because it's easy. Let $q \in B_\epsilon(p) \cap \mathcal{M}$ and let $B_\delta(q) \subset B_\epsilon(p), E_q(x) \subset E_p(x)$. Then $U_q = \{x | \bar\partial u_x \in E_q(x)\}$ and $U_p = \{x | \bar\partial u_x \in E_p(x)\}$. Then you need analysis to get the coordinate change to be smooth but once you have it, the complicated combinatorics disappears because the sheaf condition is automatic.

So then you get the Kuranishi structure. I'll define compatibilities tomorrow.

5. August 31: Lecture V

Today I said I'd give some applications, but usually there is big machinery. I want to explain one application, the most classical one. So suppose (M, ω) is a symplectic manifold and M is compact, and we consider X as $M \times M$ with the form $(\omega, -\omega)$, and L is M but as the diagonal, $\{(x,x) : x \in M\}$ and this is Lagrangian.

So the claim I want to prove is that for $H : [0,1] \times X \to \mathbb{R}$, so $\mathcal{X}_{H_t}$ is the (time-dependent) Hamiltonian vector field, and φ_H^t from X to X is, this is a standard thing in symplectic topology,

$$\frac{d\varphi_H^t}{dt} = \mathcal{X}_{H_t} \circ \varphi_H^t.$$

We assume $L \pitchfork \varphi_H^1(L)$

Theorem 5.1 (Arnold conjecture). $H(L \pitchfork \varphi_H^1(L)) \geq \operatorname{rk} H(L, \mathbb{Q})$.

So if you instead take M compact symplectic and $\overline{H} : M \times (0,1) \to \mathbb{R}$ we get $\varphi^1_{\overline{H}} : M \to M$. Then

$$\mathrm{Fix}_{\varphi^1_{\overline{H}}} = \{\gamma \in M | \varphi^1_{\overline{H}}(x) = x\}$$

and with transversality, we get the slightly weaker version

Theorem 5.2. *$H(\mathrm{Fix}_{\varphi^1_{\overline{H}}}) \geq \operatorname{rk} H(M, \mathbb{Q})$.*

We proved this in some heavy way in our book, and I want to pick out the key part that comes from the machinery I've built.

So usually what one does is define Floer homology of L, the main thing is to show that a certain boundary operator squares to zero. Then you show that $HF(L,L) \cong H(L)$ and finally you show that $HF(L, \varphi_H(L)) \cong HF(L,L)$, and that's usually good enough.

In the Lagrangian case $HF(L,L)$ has some kind of trouble, which I want to explain. Let me take $\beta \in H_2(X, L; \mathbb{Z})$ and consider $\mathcal{M}_2(\beta)$. You have two boundary operators, ev_1 and ev_0, to L, and I want to define m_1^β as a map from $\Omega(L) \to \Omega(L)$, and this is something defined in the following way,

$$m_1^\beta(h) = (\mathrm{ev}_0)_!(\mathrm{ev}_1^* h, \hat{S}^\epsilon),$$

where the thing we have is the CF-perturbation we built yesterday. This is a kind of correspondence. So then we take Λ_0, the ring of series $\sum a_i T^{\lambda_i}$ where $a_i \in \mathbb{R}$ and $\lambda_i \geq 0$ and grows without bound.

Then $\delta = d + \sum_\beta T^{\beta \cap \omega} m_1^\beta$. This is the usual Floer type thing. This does not have $\beta = 0$ because that map is constant but then it can't be stable.

The important thing is that $\delta \circ \delta \neq 0$ in general. This is related to obstruction theory. You want the correct operator to do this business.

In this particular case of diagonals, you have two options that I know about to fix this defect. One is, if you have $\tau : M \times M \to M \times M$, the anti-holomorphic involution, and you try to make everything invariant with respect to τ. To see the cancellation the most delicate thing is the sign. Since we're doing this integration, we can't work over $\mathbb{Z}_2$. The sign issues are actually very delicate. There are two things: well-definedness and isomorphism. Then you may think that $\delta = d$ because the higher things look like pairs. If you try to do this with sign you find out it's not correct. For well-definedness it's okay. But in general sometimes it works and sometimes not. In front of the blackboard I can't do it.

So I'll do it another way, with so-called *bulk deformations*. We cansider $\mathcal{M}_{2,\ell}(\beta)$ [pictures]. On one side you have $L \times X^\ell$ and on the other side evaluation to L. In a similar way as before you try to get a map $\Omega(L) \otimes \Omega(X)^{\otimes \ell} \to \Omega(L)$, $q_{1,\ell}$.

So we have b which we write $\sum_i T^{\lambda_i} b_i$ for $b_i \in \Omega(X)$ closed, and all $\lambda_i > 0$. Now

$$\delta^b(h) = dh + \sum_{\beta,\ell} T^{\beta \cap \omega} q^\beta_{1,\ell}(h, b, \dots, b)$$

converges in $\Omega(L) \hat{\otimes} \Lambda_0$, where this is

$$\{\sum T^{\lambda_i} h_i\}$$

with h_i in $\omega(L)$ and each $\lambda_i \geq 0$ and increasing to ∞.

Theorem 5.3. *For $L = M$ In $M \times M$, there is a b in $\Omega(X) \hat{\times} \Lambda_+$ such that $\delta^\beta \circ \delta^b = 0$.*

Let me try to prove this, for $\delta(h) = dh + \sum T^{\beta\cap\omega} m_2^\beta(h)$.

So we have
$$dm_1^\beta + m_1^\beta d + \sum m_1^{\beta_1} m_2^{\beta_2}.$$
Our Stokes theorem is $d(\mathrm{ev}_0)_! h + (\mathrm{ev}_0)_! h = \mathrm{ev}$, well, actual Stokes is
$$(d \circ f_!)h + (f_! \circ d)h = (f|_{\partial M})h.$$
So for instance, if the target is a point you get the usual Stokes theorem. You can prove this if you started with manifolds and orbifolds.

This is one kind of important ingredient to the theory of integration. So what I want to prove is the formula I wrote before. You have the first terms are the bracket. Then the first operations $dm_1^\beta + m_1^\beta d$ is an obstruction. This is Stokes. I want to explain another formula, a comparison formula. There is something like a composition formula, I want to, the day before yesterday I defined integration of differential forms and pushouts. One theorem we need to prove is Stockes, the other is this composition formula.

So you have N_1 and N_2 and N_3 fitting into this diagram, with f_2 and q_2 submersions:
$$\begin{tikzcd} & \mathcal{M}_1, S_{1,\epsilon} \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & \mathcal{M}_2, S_{2,\epsilon} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ N_1 & & N_2 & & N_3. \end{tikzcd}$$
So we can take the pullback $\mathcal{M}_1 \times_{N_2} \mathcal{M}_2$. Then I have correspondences $\mathcal{M}_1, f_1, f_2)$ from N_1 ot N_2

[I can't read the board, skipping]

Then the theorem is that $(\Xi_2 \circ \Xi_1)_* = (\Xi_2)_* \circ (\Xi_1)_*$. The proof is not hard if correctly stated. You can do manifold theory, use partitions of unity to reduce to one chart, and then get to manifolds and submersions, where it's just Fubini's theorem.

So now we have Stokes and composition. You see that this is the composition formual in our equation for zero, so that $m_1^{\beta_1} \circ m_1^{\beta_2}$ is exactly this correspondence $\mathcal{M}_2(\beta_1) \times_L \mathcal{M}_2(\beta_2)$, this yields the compositions. So this is rather nice. You see many similar situations, you want some algebraic formula about, you have something to be a chain map and some leftover, and you can often translate this equality into something about moduli spaces.

So the first term is boundary and the second term is fiber product. This was what we wanted to do yesterday, we wanted families of Kuranishi structures and CF perturbations that realize this kind of picture.

But the problem is that this picture is not correct in general. After you do this work, the picture proof comes out to really be a proof. Even as a picture proof there is a famous trouble. [pictures]. So if $(\mathrm{ev}_0)_![\mathcal{M}_1(\beta)]$ [unintelligible]you will have trouble.

I want to eliminate this bad effect with this bulk deformation.

If the homology of $(L, \mathbb{Q})$ injects to the homology of $X, \mathbb{Q}$, then you can use b to eliminate this term.

This is a general principle, but you have a kind of obstruction, this actually lies in a kernel of this map, and so this is injective in our case and you can do it. I'm going to do it more directly.

In our case, h is in $\Omega(\Delta)$ and $dh = 0$ so there is an $\hat{h} \in \Omega(M \times M)$ with $i_* \hat{h} = h$. For us, $\hat{h} = \mathrm{pr}_1^* h$.

I want to do this kind of deformed thing $m_1^b(h) = dh + \sum \frac{1}{\ell!} q_{1,\ell}^{\beta}(h, b, \ldots, b)$.

So this is related to the moduli space $\mathcal{M}_{2,\ell}(\beta)$, so I'll consider its boundary. The boundary is described by the following [pictures].

So with a similar business you can prove the following things. Maybe I should write

$$\delta_1^b(h) = \sum T_{!\ell}^{\beta\cap\omega}(h, b, \ldots, b)$$

so $m_1^b \circ m_1^b$ [sic].

One can check that if $\sum_\ell \frac{1}{\ell!} q_{0,\ell}(b, \ldots, b) = 0$, then $\delta^b \circ \delta^b = 0$.

So you have $\mathcal{M}_{1,\ell}^{\beta}$, you have evaluations to X^ℓ and L.

The claim is that there exists b so that $\sum_{\ell=0}^{\infty} \frac{1}{\ell!} q_{0,\ell}(b, \ldots, b)$.

[illegible]

The key observation is that the $q_{0,1}^0$ is just the pullback along the inclusion of L in X.

So we use this fact and surjectivity to construct this thing.

Let G_0 be the set of $\omega \cap \beta$ such that $\mathcal{M}(\beta) \neq 0$. Gromov compactness says that G_0 is discrete. Then G is the monoid (additive) generated by G_0, this is again discrete. Then we write $G = \{\lambda_1, \ldots, \lambda_n, \ldots\}$ but we only need to consider exponents in G going forward.

I want to construct $b = \sum_1^\infty T^{\lambda_i} b_i$.

Take $b_{(k)}$ to be $\sum T^{\lambda_i} b_i$. Then I want to prove that $\sum q_{0,\ell}(b_{([unintelligible])}), \ldots, b_{([unintelligible])})$ is zero modulo T_{k+1}^{λ}. I want to do this inductively on k.

If $k = 1$ then $b(0) = 0$ so $\sum_\ell (0, \ldots 0) = m_{0,1}$. This is trivial module T_1^λ. So now for the $k-1$ step, we have $\sum q_{0,\ell}(b_{k-1}, \ldots, b_{r-1})$ is hT^{λ_k} modulo T^{λ_k+1}.

[not following]

So now the last thing I want to prove is that the Floer homology is isomorphic to the homology of the manifold itself. This is actually routine if you know Floer homology. I won't go to that place then but the thing I want to prove is the following. I define δ^b as $d + \sum T^{[unintelligible]}/\ell! q_{1,\ell}(\cdot, b, \ldots, b)$.

So we showed $\delta^b \circ \delta^b = 0$

Theorem 5.4. *$(\Omega(M), \delta^b)$ and $(\Omega(M), d)$ are chain homotopy equivalent.*

Maybe this seems a little disappointing at first, but you can calculate this first one.

So how to prove it. The general argument is the following thing. In general there is a spectral sequence from $H(L)$ and ends at $HF(L, L)$. They are related by a spectral sequence, and this theorem says that this spectral sequence degenerates, there are no differentials. I want to do something more primitive by constructing an explicit chain map. This chain map is written in our paper about anti-holomorphic involutions. I want to do some other things, I want to take $\mathcal{M}_{1,\ell+1}(\beta)$, and evaluate on one side to $L \times X^\ell$ as "inputs" and now evaluate to X as output. Before I did this only to L. I define p as a map $\Omega(L) \otimes \Omega(X)^{\otimes \ell} \to \Omega(X)$. This goes $h, g_1, \ldots, g_\ell$ to $p(h, g_1, \ldots, g_\ell)$.

So $b \in \Omega(X)$ is something we take, and I want to think of p^b as $\sum p(h, b, \ldots, b)$, and I want to take $i^* \circ p^b$. The claim is that this one gives the chain homotopy equivalence.

So I'll take $dp^b - p^b d$, this is induced by the boundary of the moduli spaces, $\partial\mathcal{M}_{1,\ell+1}(\beta)$. So this is routine, you look at the boundary and then a picture proof is a proof. There are basically two cases. One interior marked point is different from other boundary marked points. All others are inputs. So we have to remember which one is the input. Here is one picture, where the boundary and output are in the same component. The other option they're in different components.

Then the claim is that the ones where they're in the same component do not contribute. One of them is $q_1(b,\ldots,b)$, so you only consider the other picture. You have in the other picture one which is $q_{1,\ell}(h,b,\ldots,b)/\ell!$. The second component is p^b. So this gives

$$p(q(h,b,\ldots,b),b,\ldots,b).$$

So what we calculate is that $dp - pd$ is $p\circ m^{[unintelligible]}$ so $dp = p(d+\sum m^\beta)$ so $dp + p\delta$ is zero so p is a chain map from (Ω,δ^b) to (Ω,d).

I want to show that it induces an isomorphism on homology. [pictures]

I want the leading order term, over a power series ring the leading order being a chain homotopy equivalence implies the whole thing is a chain homotopy equivalence. So the leading order term is $\mathcal{M}_{1,1}(0)$, so you pull back from L to L and then push out from L to X. The pushout from L to X is somehow submersive. This is not submersive so you need to do a perturbation to get a pushout. There are many other ways but you take the following thing.

What do we do? We take this Kuranishi neighborhood, this normal bundle of L in X, an ϵ-ball in this normal bundle, I want to cover this by one chart, a neighborhood, a the normal bundle, and we have $\pi : U \to L$, and $\mathcal{E}$ is the pullback of the normal bundle under projection. Then the section is $x \mapsto (x,x)$, over the fiber you can take the same point of the same fiber. Then the evaluation map is a submersion.

Now we recall what we are doing on the CF perturbations, you can take W, maybe, there are many ways, you, I think the pushout, with respect to this evaluation map, takes h to a modifier smoothing of h and δ_Δ. You use this perturbation, and make it a smooth form near L, perturb the delta function a little bit.

[pause]

So I made a mistake, p^b_β is a map from $\Omega(L)$ to $\Omega(X)$, and then I want to take $\mathrm{pr}_!$ to $\Omega(L)$. Then the formula I wrote is a bit wrong. You should not take i^*. Let's go back to the chain map thing. So we used the correspondence to define the map to $\Omega(X)$ and then I was supposed to project to L. Now I want to go back to the proof that this was a chain map. The formula was—

[pause]

$$d\hat{p}^b_\beta + \sum p^b_{1,\beta}\circ m^b_{1,\beta} + \hat{p}_p d = 0.$$

So p_β is something like the sum $p_{\beta,1}$—I want to write the composition $\bar{p}_\beta$. So I want to write

$$p_\beta = \sum \bar{p}_{\beta_1}\circ\cdots\circ\bar{p}_{\beta_k}.$$

[some confusion]