KENJI FUKAYA: LAGRANGIAN CORRESPONDENCE AND A-INFINITY FUNCTOR CENTER FOR GEOMETRY AND PHYSICS (POHANG)

GABRIEL C. DRUMMOND-COLE

1. August 12

I want to talk something about constructions of functors. Let me recall what everybody knows about homology theory. You have a space X and you associate this group $H(X)$, and to a continuous map $f : X \to Y$ you get a homomorphism $f_* : H(X) \to H(Y)$.

You also have something called quantum cohomology. When X is a symplectic manifold, you have the quantum cohomology ring $QH^*(X)$. It is no problem that when you have a symplectic diffeomorphism, this leads to an isomorphism of quantum cohomology. But if you have any morphism, you don't get the naive guess that there is $f^* : QH(Y) \to QH(X)$. There were several attempts to make this more functorial. In the symplectic world there is a classical proposal by Weinstein. Now complex manifolds are nice, there are morphisms, but symplectic morphisms are not well-understood. He said that if you have X_i, ω_i symplectic, that $L \hookrightarrow -X_1 \times X_2$, where $-X_1$ is $(X_1, -\omega_1)$, if you have such a Lagrangian submanifold you should regard this as a morphism. This is nice for many reasons. If you have $L_{12} \subset -X_1 \times X_2$ and $L_{23} \subset -X_2 \times X_3$, then you can take the fiber product, and if the fiber product is transversal (this is a kind of nasty assumption necessary to make the composition well-defined) then L_{13} is an immersed Lagrangian in $-X_1 \times X_3$. So you can kind of compose. But then you need to go to the category of immersions, not necessarily embeddings.

The proposal by Werheim–Woodward was to construct something like a category, from (X, ω) with Lagrangian correspondence to try to construct a functor to all A_∞ categories. This is called the Weinstein category. What they could do, they considered monotone Lagrangians and monotone compositions, that's what they could handle.

I want to realize this kind of project in complete generality.

This has a natural mirror. Consider X_1 and X_2 as a complex manifold. Let $\mathcal{E}$ be an object of the derived category of coherent sheaves on the product $\mathbb{D}(X_1 \times X_2)$. For example, if f is a holomorphic map $X_1 \to X_2$ then the graph of f is a complex submanifold and gives an object of this derived category.

Then we have this classically established morphism, $\mathbb{D}(X_1) \to \mathbb{D}(X_2)$, the functor $F_{\mathcal{E}}$, then you take $R(\pi_2)_!(\pi_1^*(C^*) \boxtimes \mathcal{E})$. The tensor product and push down are derived, you have the tors and things. This is called the Fourier–Mukai category.

So first our guess is that our objects are immersed Lagrangian submanifolds. For this business it's essential to go beyond the embedded case. It will be necessary to take immersions to make composition well-defined. I want to cook up the functors. I have L_{12} in $-X_1 \times X_2$ and L_1 in X_1, and I want to cook up L_2 in X_2, and a natural candidate is the fiber product $L_1 \times_{X_1} L_{12}$. If the L are Lagrangian and the fiber product is transversal, then L_2 is an immersed Lagrangian submanifold of X_2.

This is a kind of naive version. Then we should ask how this is related to Floer homology. So far we have geometric things. How does this work with Floer theory? I want to recall immersed Floer theory, due to Akahi–Joyce. Let X be a symplectic manifold. Take V a real oriented vector bundle over $X_{[3]}$, the 3-skeleton. An immersed Lagrangain submanifold is $i_L : \tilde{L} \to X$, an immersion of half dimension where the symplectic form pulls back to zero. Now σ is a spin structure on $T\tilde{L} \oplus i_L^*(V)$ on $\tilde{L}_{[3]}$. I want to assume that $\tilde{L}$ is self-clean. What does that mean? Take the fiber product $\tilde{L} \times_X \tilde{L} = L(+)$. What is $L(+)$? It's $\tilde{L} \sqcup \bigcup L(a)$. [pictures]. Here $\tilde{L}$ sits inside $\tilde{L} \times_X \tilde{L}$ as the diagonal (x,x) such that $x \in \tilde{L}$. The assumption is that $L(a)$ is C^∞ and $T_{p,q}L(a) = d_p i(T\tilde{L}) \cap d_q i(T\tilde{L})$. But (p,q) and (q,p) are different points in the fiber product. So $CF(L) = \Omega(L(+))\hat{\otimes}\Lambda_0$, where Λ_0 is the Novikov ring $\{\sum a_i T^{\lambda_i}$ for $a_i \in \mathbb{R}$ and $\lambda_i \geq 0$ and increasing to ∞.

Theorem 1.1. *There are $m_k : CF(L)^{\otimes k} \to CF(L)$ satisfying the A_∞-relations.*

Definition 1.1. L is *unobstructed* if there is $b \in CF(L) \otimes_{\Lambda_0} \Lambda_+$ with $\sum m_k(b, \ldots b) = 0$. Here $\Lambda_+ = \{a_i T^{\lambda_i} : \lambda_i > 0\}$.

The main theorem is

Theorem 1.2. *If L_1 and L_{12} are unbostructed then L_2 is unobstructed.*

Just immersed Lagrangians, you might not be able to do Floer homology. You really need unobstructed.

Remark 1.1. Let L be immersed in $\mathbb{C}^n$, self transversal nad unobstructed. Then the number of self-intersections is at least half the total Betti number.

This is not correct in general, without the unobstructed condition.

One can say the following thing. Take Hamiltonian perturbations of L_{12}. This can be very complicated. The topology of L_2 can change. The theorem says that still it is unobstructed. Further, its Floer homology is independent of the Hamiltonian perturbation.

Now we can say the following things. The Weinstein category does not seem to be good enough for Floer theory. So I want to propose the *unobstructed Weinstein category* whose objects are (X, ω, V), where V is the bundle on V.

Ah, I should have said, if L_1 is V_1-relatively spin and L_{12} is $TX_1 \oplus V_1 \oplus V_2$-relatively spin, then L_2 is V_2-relatively spin. I need these assumptions to state the theorem. I left that out.

So f is a symplectic diffeomorphism and L_f is the graph of f. Then L_f is diffeomorphic to X_1 but may not be spin. But it's always TX_1-relatively spin. With Yong-geun and the others, we studied something like Floer homology on $X \times X$. At some point we realized that something was not spin, so we tried to use this kind of relative spin.

Okay, so the morphisms are L_{12} immersed in $X_1 \times X_2$ with $TX_1 \oplus V_1 \oplus V_2$ which is relatively spin, along with a bounding cochain b_{12}.

Theorem 1.3. *There is a functor F from the unobstructed Weinstein category to the category of all filtered A_∞ categories.*

This is kind of the main theorem of these talks. You have this triple, you can construct an A_∞ category.

I think $HH(F(X_1,\omega,V))$ has an open-closed map to $H_*(X_1)$, and this is covariant with respect to A_∞ functors, and you can map to $HH * F(X_2,\omega,V)$, and I claim that this square should commute.

$$\begin{tikzcd} HH_*F(X_1,\omega_1,V_1) \arrow[r] \arrow[d] & HH_*F(X_2,\omega_2,V_2) \arrow[d] \\ H_*(X_1) \arrow[r] & H_*(X_2) \end{tikzcd}$$

Let me consider i_L, the immersion $\tilde{L} \to X$, and $CF(L)$ which is $\Omega(\tilde{L} \times_X \tilde{L})\hat{\otimes}\Lambda_0$. So $\tilde{L} \times_X \tilde{L}$, let me write it as the diagonal components $\tilde{L}$ along with $\bigcup L(a)$ for $\vec{a}$ which is $(a_0,\dots a_k) \in \mathcal{A}^k$. Now let $\circ M_{\vec{a}}(L,E)$ be the set of $(D^2,\vec{z},U,\gamma)$ where $\vec{z}$ is a cyclically ordered set in the boundary of D^2, $U:(D^2,\partial D^2) \to (X,L)$ is holomorphic and $\gamma : \partial D^2 \backslash |\vec{z}| \to \tilde{L}$ and $i_L \circ \gamma = U$ on $\partial D^2 \backslash |\vec{z}|$. Finally, when z approaches z_i, you get $\gamma(z_{i+1})$ is in $L(a_i)$. [picture]. The last condition is that $\int U^*\omega = E$.

Now M can be compactified, and you get a map $M_{\vec{a}}(L,E) \to L(a_0) \times \cdots L(a_k)$. So you have projections

$$\begin{tikzcd} & M_{\vec{a}}(L,E) \arrow[dl] \arrow[dr] & \\ L(a_1) \times \cdots \times L(a_k) & & L(a_0) \end{tikzcd}$$

So you pull back and push out by integration along the fibers, and you get $m^E_{\vec{a}} : \Omega(L(a_1)) \times \cdots \times \Omega(L(a_k)) \to \Omega(L(a_0))$.

Then $m_k = \sum_{\vec{a}} T^E m^E_{\vec{a}}$.

[pictures.]

Now I want to go to an A_∞ category from the A_∞ algebra. Let me assume everything is oriented. It's better to use a finite set of self-clean Lagrangian submanifolds, you have $\mathcal{L}_1 = \{L_{1,i} | i = 1,\dots,N\}$. You just take the unions $L = \cup L_{1,i}$, and assume that this one is self-clean. This just means that you have, well, each one is self-clean and they have pairwise clean intersections. To go to the A_∞ category you can think of this as a single Lagrangian. So you get $CF(L)^{\otimes k} \to CF(L)$, and you see that $CF(L)$ is the sum of all $CF(L_i)$ along with $\bigoplus \Omega(\tilde{L}_i \times_{X_1} \tilde{L}_j)\hat{\otimes}\Lambda_0$.

So you have $CF(L_i,L_j) = \Omega(\tilde{L}_i \times_{X_1} \tilde{L}_j)\hat{\otimes}\Lambda_0$. So $CF(L)$ is just $\bigoplus CF(L_i) \oplus \bigoplus CF(L_i,L_j)$. Then m_k for L is $m_k : CF(L_0,L_1) \otimes \cdots \otimes CF(L_{k-1},L_k) \to CF(L_0,L_k)$. This gives us something called an A_∞ category.

This step is not actually so difficult.

Now let me make a more precise statement. Suppose you have X_1 with ω_1, and V_1. Take $\mathcal{L}_1$, a finite set of immersed Lagrangian submanifolds. Take $(-X_1 \times X_2, -\omega_1 + \omega_2, TX_1 \oplus V_1 \oplus V_2)$, and then $\mathcal{L}_{12}$ a finite set of immersed Lagrangian submanifolds. Assume for all $L_{1,i}$ and all $L_{12,j}$ that this fiber product $L_{1,i} \times_{X_1} L_{12,j}$ is transversal. This you can achieve by perturbing slightly.

Now I want to consider $\mathcal{L}_2$, a finite set of a Lagrangian submanifold which contains $L_{1,1} \times_{X_1} L_{12,j}$.

From $\mathcal{L}_1$ you have this filtered A_∞ category whose objects are in $\mathcal{L}_1$. This A_∞-category is inconvenient, because this object may be obstructed. There are general constructions using bounding cochains. I'll go back to geometry soon, but let me do some algebra.

Let C be a filtered A_∞ category. What does it mean? You have a set of objects and then c_1 and c_2 are objects and $C(c_1, c_2)$ is a Λ_0-module. You have $m_k : C(C_0, C_1) \otimes \cdots \otimes C(C_{k-1}, C_k) \to C(C_0, C_k)$, which satisfies the A_∞ relation. So you want to define bounding cochains here because you have m_0 here. You have another A_∞ category, with $m_0 = 0$. The objects of $\bar{C}$ is a pair (c, b), where $c \in C$ and $b \in C(c, c) \otimes \Lambda^+$ which satisfies the Maurer-Cartani equations. I just want to start from the A_∞ category. When m_0 is nonzero then m_1^2 is nonzero so you can't define Floer homology. The morphisms are just the same, but m_k is different, $\bar{m}_k(x_1, \ldots x_k)$ is $m(e^{b_0} x^1 e^{b_1} \cdots x_k e^{b_k})$. where x_k is in $\bar{C}((c_{k-1}, b_{k-1}), (c_k, b_k))$. Here e^b is $1 + b + b \otimes b + \cdots$. So an object is a pair, and the composition uses b insertions. Then $\bar{m}_0(1) = m(e^b) = 0$.

We call such a thing strict. This is a pure algebraic construction. What we do is the following. We have X_1 and $\mathcal{L}_1$ the finite set of Lagrangians. Then you have $-X_1 \times X_2$, and $\mathcal{L}_{12}$ and $\mathcal{L}_2$ which contains $\{L_{1,i} \times_{X_1} L_{12,j}\}$.

Now you have $\mathcal{L}_1$ an A_∞ category whose objects are $L_{1,i} \in \mathcal{L}_1$ and $F(\mathcal{L}_1)$ leads to $\bar{F}(\mathcal{L}_1)$, an A_∞ category with $m_0 = 0$.

A theorem that I'll prove maybe next time is that

Theorem 1.4. *There is a filtered A_∞ bifunctor $\bar{F}(\mathcal{L}_1) \times \bar{F}(\mathcal{L}_{12}) \to \bar{F}(\mathcal{L}_2)$.*

Maybe I think today it's better to state the things I want to prove in the first three lectures. This was proven in the assumption that everything was embedded and monotone.

Now I want to say about the compositions. I've given the space of morphisms. The notion of A_∞ bifunctor should be defined but I'll do it later, postpone it because I've done too much algebra.

Now the next thing is composability. Let me remind you that $\mathcal{L}_{12}$ is a set of Lagrangian submanifolds of $-X_1 \times X_2$. I want to choose another finite set $\mathcal{L}_{23}$ of Lagrangian submanifolds of $-X_2 \times X_3$. Then something I want to assume is the following. Assume that $L_{12,i} \in \mathcal{L}_{12}$ and $L_{23,j} \in \mathcal{L}_{23}$, and that $L_{12,i} \times_{X_2} L_{23,j}$ is transversal. Then the next theorem is, well, then $\mathcal{L}_{13}$ is a finite set of Lagrangian submanifolds of $-X_1 \times X_3$, and I want to say that this contains all of these such fiber products. You consider the finite set of morphisms, and that transversality is okay and then a finite set containing the compositions.

Theorem 1.5. *There is a bifunctor $\bar{F}(\mathcal{L}_{12}) \times \bar{F}(\mathcal{L}_{23}) \to \bar{F}(\mathcal{L}_{13})$.*

For example, if L_{12} and L_{23} are unobstructed, then so is their fiber product. The unobstructed Weinstein category is thus a kind of [unintelligible]category.

I can probably prove this in generality, but let me write a weaker version first. Let me remind that we have $\mathcal{L}_1$, $\mathcal{L}_{12}$, and $\mathcal{L}_2$. Then you consider $\mathcal{L}_3$ which contains all the things from $\mathcal{L}_2$ as well as those from $\mathcal{L}_1$.

Theorem 1.6. *There is a commutative square*

$$\begin{tikzcd}
\bar{F}(\mathcal{L}_{12}) \times \bar{F}(\mathcal{L}_{23}) \arrow[r] \arrow[d] & \bar{F}(\mathcal{L}_{13}) \arrow[d] \\
Func(\bar{F}(\mathcal{L}_1), \bar{F}(\mathcal{L}_2)) \times Func(\bar{F}(\mathcal{L}_2), \bar{F}(\mathcal{L}_3)) \arrow[r] & Func(\bar{F}(\mathcal{L}_1), \bar{F}(\mathcal{L}_3)
\end{tikzcd}$$

[some discussion.] What I can prove is, if I have L_{12}, b_{12} and L_{23}, b_{23}, then we get (L_{13}, b_{13}), and what we can prove, is well (L_{12}, b_{12}) yields $W_{(L_{12},b_{12})} : F(\mathcal{L}_1) \to F(\mathcal{L}_2)$ and then

$$W_{(L_{23},b_{23})} \circ W_{(L_{12},b_{12})}$$

is homotopic to $W_{(L_{13},b_{13})}$.

The next thing is the following thing. Suppose that L_{12} and L'_{12} are homotopic. Then $\varphi_t : -X_1 \times X_2 \to -X_1 \times X_2$ is a Hamiltonian isotopy. Then somehow, what I want to say is the following. First of all, the isotopy, b_{12} gives b'_{12}, this is general theory of Lagrangian submanifolds. What I want to claim is that $W_{L_{12},b_{12}} \sim W_{L'_{12},b'_{12}}$ over $\Lambda = \Lambda_0[T^{-1}]$. If you look at Joyce's paper about immersed Lagrangian Floer theory, there's one thing that's difficult to understand. If you have these two Lagrangians, then $HF(L, L')$ and $HF(\varphi(L), \varphi'(L'))$ are equivalent over Λ. But for immersed things, you can do isotopy locally. You have $\tilde{L} \times [0,1] \to \mathbb{R}$, and things are quite delicate. If you use this Hamiltonian [picture], it's hard to understand what to do with the Floer homology. If the Hamiltonian isotopy, well, now you don't know that L_2 is isotopic to L'_2. They might not even be diffeomorphic. We can still prove the theorem about homotopy equivalence of the W.

So suppose you have another $L \subset X_2$ with b, then we can show that $HF(W_{(L_{12},b_{12})}(L_1, b_1), (L, b')) \otimes \Lambda$ is isomorphic to $HF(W_{(L'_{12},b'_{12})}(L_1, b_1), (L, b)) \otimes \Lambda$. There's something delicate related to the Künneth theorem.

[some discussion].

There is a notion of Lagrangian cobordism. In the monotone situation, cobordant Lagrangians have the same Floer homology. It seems very likely that we can extend that story in the unobstructed category. If the cobordism admits an extension of the bounding cochain, then we expect the Floer homology is the same.

2. August 14

[This was a national holiday and I missed the lecture.]

3. August 17

Suppose you have a symplectic manifold X_1. You cook up this category $\bar{F}(X_1)$, that has objects (L, σ, b), where L is an *immersed* Lagrangian, σ is a relative spin (I talked about this last time) and b is an element in the Floer chain complex of L. Then for $\mathcal{L}_{12}$ which is $L_{12}, \sigma_{12}, b_{12} \subset -X_1 \times X_2$, we want to cook up an A_∞ functor $\mathcal{W}_{\mathcal{L}_{12}} : \bar{F}(X_1) \to \bar{F}(X_2)$. I want to show that this is compatible with composition of Lagrangian correspondence.

Essentially what you want to show is, suppose L_1 has a Maurer–Cartan solution b_1 and L_{12} has a Maurer–Cartan solution b_{12}, we want to find b_2 which is a Maurer–Cartan solution. There are a couple of ideas. Here's one, due to [unintelligible].

Consider a strip in $\mathbb{C}$, the strip with $Im\ z$ between 0 and 1. You want a holomorphic map from u_1, the strip, to X_1, from u_2, the space above the strip, to X_2, from the real line to L_1 and from the boundary between them to L_{12}. [pictures, confusing].

The way I want to cook up this functor is with a 3-step construction. Then I have $\bar{F}(X_1)$ and $\bar{F}(X_2)$, and I take the Yoneda embedding of $\bar{F}(X_2) \hookrightarrow Func(\bar{F}^{op}(X_2), ch)$. It's very natural to construct a "Künneth" functor $\bar{F}(X_1) \to Func(\bar{F}(X_2)^{op}, ch)$. Then there are two things you need to prove. Consider the image of the Yoneda functor $Rep(\bar{F}(X_2)^{op}, ch)$. The second theorem says that the Künneth functor lifts to the representables. The Yoneda theorem says that the map to representables is an isomorphism, so you have an inverse function θ to the Yoneda embedding. Then you can use this lift of the Künneth functor followed by a homotopy inverse functor. In this way we can bypass the complicated thing. You need to prove many things are functorial if you do this by hand. But the Künneth functor has nice functorial properties, so it's easier if you do it this way.

I want to define the notion of an A_∞ bifunctor. Künneth is delicate in A_∞ categories. What is the tensor product of A_∞ algebras? This is not trivial, so the Künneth theorem is a bit complicated. An A_∞ category has a set of objects, and for two objects c_1 and c_2, we have $\mathcal{C}(c_1, c_2)$ a Λ_0-module, where $\Lambda_0 = \sum a_i T^{\lambda_i}$ as before. Then $m_k : B_k(c, c') \to C(c, c')$, where $B_k(c, c') = \bigoplus \bigotimes \mathcal{C}(c_{i-1}, c_i)$. In this setting, there is no m_0. There are some cases where we want that. Let me say C is curved if $m_0 \neq 0$.

I want to explain A_∞ bifunctors. Suppose you have $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ which are A_∞ categories. Let $F_{ob} : ob(\mathcal{C}_1) \times ob(\mathcal{C}_2) \to ob(\mathcal{C}_3)$. Then

$$F_{k_1,k_2} : B_{k_1}\mathcal{C}_1(c_1, c_1') \otimes B_{k_2}\mathcal{C}_2(c_2, c_3)' \to \mathcal{C}_3(F_{ob}(c_1, c_2), F_{ob}(c_1', c_2')).$$

Note that $B\mathcal{C} = \bigoplus B_k(c, c')$ has a coalgebra sturcture $\Delta : B\mathcal{C} \to B\mathcal{C} \otimes B\mathcal{C}$, but lands in the subset of composable pairs. Then $B\mathcal{C}$ also has $\hat{d}$ where $\hat{d}(x_1 \otimes \cdots \otimes x_k) = \sum x_1 \otimes \cdots m(x_i, \cdots x_j) \otimes \cdots \otimes x_k)$. Then being A_∞ is the same as saying that $\hat{d}\hat{d} = 0$.

So $\hat{F} : B\mathcal{C}_1 \otimes B\mathcal{C}_2 \to B\mathcal{C}_3$, which is kind of a cohomomorphism. The tensor product of coalgebras is a coalgebra. If you just compose with the projection to $\mathcal{C}_3$ you get F_{k_1,k_2}.

This F is an A_∞-bifunctor if and only if $\hat{F}\hat{d} = \hat{d}\hat{F}$.

From this definition you can easily prove this lemma. Consider the A_∞ bifunctor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$, that's the same as a functor $\mathcal{C}_1 \to Func(\mathcal{C}_2, \mathcal{C}_3)$. I don't want to define, but there's also a similar notion of a trifunctor.

Now we have the algebraic preliminary. What do we get in the following thing? What about our Künneth. We have $-X_1$ and X_2, two symplectic manifolds. The claim is that

Theorem 3.1. *There is an A_∞ trifunctor $\bar{F}(-X_1) \times \bar{F}(X_2)^{op} \times \bar{F}(-X_1 \times X_2) \to ch$.*

This is what I want to call the Künneth functor. I just want to mention an application of this.

Corollary 3.1. *Using the tautological identification, you get a map $F(-X_1) \times \bar{F}(-X_1 \times X_2) \to Func(\bar{F}(X_2)^{op}, ch)$. This is something in the Yoneda codomain of $\bar{F}(X_2)$.*

I want to prove the theorem. The category version follows formally from the algebra version of this theorem. I want to prove the version where I don't have the bar, so that this is a *curved* A_∞ functor. So say you have $L_1 \subset -X_1$, $L_{12} \subset -X_1 \times X_2$, and $L_2 \subset X_2$, then you get D an A_∞ -trimodule over $CF(L_1)$, $CF(L_{12})$, and $CF(L_2)^{op}$. We proved in FOOO that if $L \subset X_1$ and $L' \subset L'$, that $CF(L, L')$ is an A_∞-left module over $CF(L)$ and an A_∞ right module over $CF(L')$. But it's easier to say that this is an A_∞ left module over $CF(L')^{op}$. So this is like this.

So you have u a map from a strip to X, whose boundary values are L and L'. The bimodule structure comes from putting marked points on the strip.

The trimodule version has one more line. You put in a vertical strip with a vertical line dividing it. U_1 is a map from the first half of the strip to $-X_1$ and U_2

from the second half to X_2. The left boundary goes to L_1, the right boundary to L_2, and the middle line has $(U_1(z), U_2(z)) \in L_{12}$. This is called a pseudoholomorphic quilt.

This looks like some different kind of moduli space. In fact, this moduli space is something equivalent to one which you already know about. Just put a strip between L_{12} and $L_1 \times L_2$, and define $U(z)$ as $U_1(z'), U_2(z)$ where $v = x + \sqrt{-1}y$ and $z' = -x + \sqrt{-1}y$. That makes the map to X_1 antiholomorphic, which means it's holomorphic to $-X_1$. So this moduli space looks slightly different, but in this simple case, you know that the moduli space is related to Künneth. This just relates Floer theory between L_{12} and $L_1 \times L_2$. You don't do too much new analysis, this case is established already.

So [unintelligible]wanted to take the limit when you let one of the strips get narrower, but if you keep the strips fixed the analysis is just the same.

Now there is a slightly different point. So now what is D? I wanted to construct, starting with L_1, L_{12}, and L_2, I wanted a chain complex with an action of all the Floer complexes.

D is $\Omega(L_1 \times_{X_1} L_{12} \times_{L_2} X_2) \otimes \Lambda_0$. This is $\Omega((L_1 \times L_2) \times_{X_1 \times X_2} L_{12}) \otimes \Lambda_0$. [Some discussion of pictures]. When the imaginary part goes to $\pm\infty$, you see a point in the fiber product. More precisely, you have the following.

You have $\mathcal{M}_{k_1,k_{12},k_2}(E, L_1, L_{12}, L_2) = \{U_1, U_2, \vec{z}_1, \vec{z}_{12}\vec{z}_2 | [unintelligible], |\vec{z}_i| = k_i, \int U_1^*(-\omega) + \int U_2\omega = E\}$. There are several evaluation maps, to $L_1^{k_1}$, to $L_{12}^{k_{12}}$, and to $L_2^{k_2}$. So you also have $ev_{-\infty}$ and $ev_{+\infty}$, both of which go to $L_1 \times_{X_1} L_{12} \times_{X_2} L_2$.

You have

$$\Omega(L_1 \times_{X_1} L_{12} \times_{X_2} L_2) \otimes \Omega(L_1)^{\otimes k_1} \otimes \Omega(L_2)^{\otimes k_{12}} \otimes \Omega(L_{12})^{\otimes k_{12}} \to \Omega(L_1 \times_{X_1} L_{12} \times_{X_2} L_2).$$

This is basically the trimodule structure, there are some other details but let me suppress them. Then the relation is that

$$\Delta x_1 = \sum x_1^c \otimes x_1^{c\prime}$$

and then

$$\sum_{c,d,e} n(n(y, x_1^c x_2^d x_{12}^e) x_1^{c\prime} x_2^{d\prime} x_{12}^{e}{}') +$$

$$+ \sum n(y_1, (\hat{d}x_1)x_2, x_{12}) + \sum n(y_1, x_1, \hat{d}x_2, x_{12}) \sum n(y_1, x_1, x_2, \hat{d}x_{12}) = 0$$

Now we have the trifunctor. Well, this is a trimodule, but it's kind of saying the same thing in a different way. So we get $\bar{F}(X_1) \times \bar{F}(X_{12}) \to Func(\bar{F}(X_2), ch)$. There's a slight difference between this and Künneth. If you don't put any marked points in, then you have the same thing. But the marked points on L_1 and L_2 are only hit by differential points on L_1 and L_2 separately, whereas in the product you can hit with arbitrary differential forms on the product. So this is like the difference between $BCF(L_1) \otimes BCF(L_2)$ and $BCF(L_1 \times L_2)$.

Maybe I finish the constructions after the break.

So L_1 is a relatively spin immersed Lagrangian with bounding chain, and L_{12}, b_{12} similarly, as an object of $\bar{F}(-X_1 \times X_2)$, so these give us $\mathcal{W}(\mathcal{L}_1, \mathcal{L}_{12}) : \bar{F}(X_2) \to Ch$. The claim is that there exists an $\mathcal{L}_2$ in $ob(\bar{F}(X_2))$ such that $Yo(\mathcal{L}_2) \cong \mathcal{W}(\mathcal{L}_1, \mathcal{L}_{12})$. Then by the inverse of the Yoneda embedding we'll get $\bar{F}(X_2)$.

The proof is two step. We have this particular geometric object $L_2^0 = L_1 \times_{X_1} L_{12}$. First, we want to show that there is a bounding cochain, and the second step is that $Yo(\mathcal{L}_2) \cong \mathcal{W}(\mathcal{L}_1, \mathcal{L}_{12})$.

So I want to prove this, and the key idea is that $D = CF(L_1, L_{12}, L_2) = \Omega(L_1 \times_{X_1} L_{12} \times_{X_2} L_2)$. The idea is to particularly choose L_2^0. Now I'll make a very simple geometric claim, that this is canonically diffeomorphic, $L_1 \times_{X_1} L_{12} \times_{X_2} L_2^0 \cong L_2^0 \times_{X_2} L_2^0$. But this is something like $CF(L_2^0)$, so that $CF(L_1, L_{12}, L_2^0)$ is isomorphic as a Λ_0-module to $CF(L_2^0)$. This $CF(L_1, L_{12}, L_2)$ is a *right* $CF(L_2)$-module. You have bounding cochains b_1 and b_{12} then this is an uncurved right $CF(L_2)$-module. Let me give a geometric rationale. So $CF(L, L')$ is a left $CF(L)$ module and right $CF(L')$-module. This means the following. So D is left C_1 and right C_2 module, then you have $n_{k_1,k_2} : B_{k_1}(C_1) \otimes D \otimes B_{k_2}(C_2) \to D$. The relations are complicated, something like

$$\sum n(x^c n(x^{c'} y z^d) z^{d'}) + n(\hat{d}x, y, z) + n(x, y, \hat{d}z) = 0.$$

So here what's the trouble? You just see this formula. Suppose you forget C_1, then D is not a right module structure. Let x be the empty set, just 1. Then

$$\sum n(n(y, z^c) z^{c'}) + n(y, \hat{d}z)$$

is missing the term $n(m_0(1), y, z)$. But we have the bounding chain b and can correct n from the left and define n^b so that $n^b(\vec{x}, \vec{y}, \vec{z}) = n(e^b x_1 e^b x_2 \cdots e^b, \vec{y}, \vec{z})$. This gives a right C_2-module structure on D. Geometrically speaking, if you cancel your right disk bubbles by b, you get an honest right module structure. You can do the same thing for a trimodule. You have L_1, L_{12}, and L_2. You can put b_1 and b_{12} in the same way. Then you get a right $CF(L_2)$-module. This is still curved, but only on the L_2-side. I want to prove that $CF(L_2)$ has a bounding cochain. After this, you can forget two of the module structures, and have a right $CF(L_2)$-module structure.

Then I'll mention a simple lemma. I like it, because it gives a way to get a bounding cochain.

Lemma 3.1. *Ret D be a right C-module, where C is curved. Say you have $\mathbf{1}$ in D such that $n(\mathbf{1}) = 0 \pmod{\Lambda}_+$, and $x \mapsto n(1, x)$ is a Λ_0-module isomorphism, then the conclusion is that there is a unique element b so that $\sum n_k(1, b, \ldots, b) = 0$. This implies that $\sum m_k(b, \ldots, b) = 0$.*

Once the statement is given, the proof is kind of obvious. The two conditions imply that you can do this by induction and it's unique.

We can actually use this lemma to obtain b. We're exactly in the situation in which our lemma can be applied. We have $CF(L_1, L_{12}, L_2)$, and we have b_1 and b_{12}, and this is a right $CF(L_2)$-module. You can see abstractly that $\Omega(L_1 \times_{X_1} L_{12} \times_{X_2} L_2^0) \cong \Omega(L_2^0 \times_{X_2} L_2^0)$ abstractly.

We can see that $L_2^0 \times_{X_2} L_2^0$ is L_2 (the diagonal) along with a bunch of other parts. Then we say $\mathbf{1} = [L_2]$. The first property $n(1) \equiv 0 \pmod{\Lambda}_+$ is obvious because $d\mathbf{1} = 0$. Higher derivatives come from positive energy.

Now I want to see the second property. You have the abstract isomorphism, and I want to say that the abstract isomorphism (up to positive energy) is realized by this multiplication.

We consider L_1 and L_{12} and L_2 in our strip, and we let one end of L_1 go to 1. We want to see the other end. I want to calculate $n(1, x) \pmod{\Lambda}_+$. If we forget the positive energy, we should get an isomorphism. The maps u_1 and u_2 are constant. What is x? It's a differential form on L_2 or $L_2(a)$. Let me pretend that this is self-transversal. Multiplication of differential forms on L_2, this marked point can move everywhere. You have a constant map, and the pullbacks and pushforwards are the obvious map. Then $\mathcal{M}$ is L_2 with three marked points. So you get the identity map modulo Λ_+. The other option is that you're switching components. [geometric reasoning] Again, forgetting positive energy, this is the identity map.

This is a proof that your L_2 is unobstructed. The rest of the proof is that this particular pair represent the Künneth functor. We not only get the b, but $\mathbf{1}$ is actually a cycle. To prove the isomorphism, you use the fact that $\sum n_k(\mathbf{1}, b, \dots, b) = 0$.

Let me consider that we have $\sum n(\mathbf{1}, b_2^0, \dots, b_2^0) = 0$. Now suppose we have $(L_2, b_2) \in Ob(\bar{F}(X_2))$. We have $HF((L_1, b_1), (L_{12}, b_{12}), (L_2, b_2))$. Then we want to show that this is equivalent to $HF((L_2^0, b_2^0), (L_2, b_2))$.

The proof that they are isomorphic uses the following diagrams. [pictures].

4. August 20

So last talk I just explained that there is a functor. You have X_1 and X_1 and I constructed a functor

$$\bar{F}(X_1) \times \bar{F}(-X_1 \times X_2) \to F(X_2)$$

and this is what we did last time. You have this "unobstructed Weinstein category" W whose objects are (X, ω, V), a symplectic manifold along with something related to its relative spin structure, and the morphisms are $L_{12} \subset -X_1 \times X_2$ with a bounding cochain b_{12}. We want a functor from W to the category of all A_∞ categories. What we did so far, well, we cooked up $\bar{F}(X, V)$. Now we have $\mathcal{L}_{12} = (L_{12}, b_{12})$ inducing $W_{\mathcal{L}_{12}} \bar{F}(X_1) \to \bar{F}(X_2)$. We need to show that this structure is compatible with composition of morphisms.

First we need to define composition operator on the unobstructed Weinstein category. If you have $L_{12} \subset -X_1 \times X_2$, you have $L_{23} \subset -X_2 \times X_3$, then $L_{13} = L_{12} \times_{X_2} L_{23}$. This fiber product might not exist. To make this a category there is some difficulty because of this. There are two ways to go around it. You can include some perturbations; then L_{13} which is not fully defined, you might get non-diffeomorphic things. One way, you hav $L_{23} \sim L'_{23}$, this is an unobstructed cobordism, you have a cobordism, you have L and L' in X, they are Lagrangian cobordant if you have $\tilde{L}$ in $X \times \mathbb{C}$, and outside a compact set it looks like $L \times \mathbb{R}$ and $L' \times \mathbb{R}$. If you assume everything is monotone, then the Floer theory of L and L' are the same. We're working on the non-monotone case. I didn't write the proof yet. If you have $\tilde{L}, \tilde{b}$ which restricts to L, b and L', b', then the Floer theory of these guys will be the same, at least with coefficients in Λ_0.

Then you can make sense of the fiber product always, because Hamiltonian isotopy is a Lagrangian cobordism. Then you get an honest category and you get an A_∞ functor defined up to homotopy. So you can go from the cobordism category of the Weinstein category to the homotopy category of A_∞ categories. Or you could work on a topological category where everything is defined on a dense subset. Things are well-defined only up to homotopy.

So that's not the main thing I want to explain. I want to explain two theorems.

Theorem 4.1. *Say L_{12} has b_{12} and L_{23} has b_{23}. Then $L_{13} = L_{12} \overset{\pitchfork}{\times}_{X_2} L_{23}$ also has a bounding cochain b_{13}*

Theorem 4.2. $W_{L_{23}, b_{23}} \circ W_{L_{12}, b_{12}} \sim W_{L_{13}, b_{13}}$.

Conjecture 4.1.

$$\begin{tikzcd}
\bar{F}(-X_1 \times X_2) \times \bar{F}(-X_2 \times X_3) \arrow[r] \arrow[d] & \bar{F}(-X_1 \times X_3) \arrow[d] \\
F_c(\bar{F}(X_1)\bar{F}(X_2)) \times F_c(\bar{F}(X_2), \bar{F}(X_3)) \arrow[r] & F_c(\bar{F}(X_1), \bar{F}, X_3)
\end{tikzcd}$$

commutes. Here F_c is the functor category.

The first line is a categorical version of the first theorem above. The bottom line should be pure algebra. The commutativity should be a better version of the second theorem, which is commutativity at the level of objects.

We'll use the tool of Y diagrams. It's a complicated quilt. Consider the following things: [picture].

The proof of the first theorem, consider a neighborhood of this puncture. Conformally change this picture [picture].

So we consider this moduli space. How do we use it? Suppose that U_i is J_i-holomorphic and energy is finite. As τ, the $\mathbb{R}$-factor, goes to $\pm\infty$, you get U_1, U_2, U_3 go to a point in $L_{12} \times_{X_2} L_{23} \times_{X_3} L_{31}$, contracted over X_1. This is $L_{13} \times_{X_1 \times X_3} L_{13}$.

I proved last time that b_1 and b_{12} existing proved that there was a bounding cochain b_2 and this proof will be similar. You have evaluation maps

$$\begin{tikzcd}
 & \mathcal{M}^E(L_{12}, L_{23}, L_{13}) \arrow[dl, "\tau\to-\infty"'] \arrow[dr, "\tau\to\infty"] & \\
L_{13} \times_{X_1 \times X_3} L_{13} & & L_{13} \times_{X_1 \times X_3} L_{13}
\end{tikzcd}$$

So put marked points and you get a map $CF(L_{13}) \times CF(L_{12}) \times CF(L_{23}) \times CF(L_{13}) \to CF(L_{13})$. This is a map n_{k_1,k_2,k_3}, so now we take $n_k^b(y, x_1, \dots, x_k)$ to be

$$\sum n^k_{k_1,k_2,k_3}(y, b_{12}^{k_{12}}, b_{23}^{k_{23}}, x).$$

[picture]

This gives us a right $CF(L_{13})$ modele structure on $CF(L_{13})$. Then there is a unique b_{13} so that $\sum n_k(\mathbf{1}, b_{13}, \dots, b_{13}) = 0$. This is from the algebraic lemma I explained last time. We can then cook up the bounding cochain.