MICROLOCAL CATEGORIES
DMITRY TAMARKIN

GABRIEL C. DRUMMOND-COLE

Maybe we can start already. So I'm going to discuss the following thing. Suppose I have a compact symplectic manifold M. We have the so-called Fukaya category. Its objects are Lagrangian submanifolds, and if two intersect in a nice way, you can construct a Hom space with a basis of the intersection points. You have a differential and composition maps, and for all this to be constructed, you need to count—

[change of room]

To construct a category without pseudoholomorphic disks was the motivation here. One of our main ingredients will be the so-called microlocal analysis. This is because microlocal analysis does the following. If you have X, you can look at a story on X under a microscope, it becomes a story on T^*X. This is so, so, so, local. Normally you concentrate on a neighborhood of a point, and microlocal means you look only at neighborhoods with a certain direction.

It was discovered that solutions to differential equations sometimes behave nicely in some directions. So if $\mathscr{F}$ is a sheaf on X, then we can look at the singular support of $\mathscr{F}$, which is in T^*X, and we will discuss the meaning of this. Then it means, for example, well, T^*X is symplectic, and we can consider Lagrangians inside of it, and consider objects microsupported on the Lagrangian A. This gives a similar category to the Fukaya category. If your Lagrangian is nice, then the objects supported on this are small. For example if it's the zero section, then you get locally constant sheaves. But why doesn't this work immediately? The singular support of $\mathscr{F}$ is a conic subset, which means that, well, T^*X is a vector bundle, so you have a dilation, A is conic if $\lambda A \subset A$ for a positive number λ.

It turns out that conic subsets are the only ones you can see in microlocal analysis. We'd like to see all subsets. This can be done by a simple trick: consider sheaves instead on $T^*(X \times \mathbb{R})$. If I look at the fiber of this cotangent bundle, the fiber will be $T_x^*X \times \mathbb{R}$. A nonhomogeneous subset A, something that is not conic, you can make it into a cone in $T_x^*X \times \mathbb{R}$. [Picture].

Now all sheaves on X can be supported here and we can use our microlocal analysis. For some examples you get something that is more or less the same as the Fukaya category. In order to match, it should be defined over a Novikov ring. You have this enriched structure more or less for free, because A becomes invariant under translation along the $\mathbb{R}$ factor of the base. The conification doesn't depend on the point in the base. So then from that you can get a Novikov ring structure.

Then we'll have some picture which is similar to the Fukaya category. This is all nice except that it only works for the cotangent bundle. Then the next question is how to make it work for an arbitrary symplectic manifold, or at least compact with integral period symplectic form (or rational? I missed it).

So the idea is just, if you have $M = \cup U_a$, a symplectic manifold covered by Darboux charts, $U_a \hookrightarrow T^*\mathbb{R}^{(\dim\ M/2)}$, and then you can take the subset of all objects supported on U_a. So you can, more precisely, look at sheaves on $\mathbb{R}^{\dim\ M/2} \times \mathbb{R}$ microsupported on the conification of U_a. Here it's a rather unusual situation. Usually you expect to get a sheaf of categories on M. Here it's not going to be the case. First of all, if it's a sheaf of categories, we lose hope that it will be the Fukaya category, which cannot be localized over M. We'll have a presheaf of categories, but it won't satisfy the sheaf condition. So the problem becomes nontrivial. There are two steps to solve it. If you take the Fukaya category and divide out by the maximal ideal of the Novikov ring, so there should be a classical limit which will be a sheaf of categories. This is more or less what's going on. You need to define what it means to take the classical limit, and then you have a sheafification, and so you get, you can do everything on the classical level, and you get a sheaf of categories on M, and then you would like to get back to the whole story of a Novikov ring. We'd like to lift from the quotient to the whole Novikov ring, and this is the problem solved by deformation theory. This is nontrivial and involves the operad of little three-disks.

So consider $F \times B \to M$. This should cover M, and hopefully you can promote this to a Lagrangian correspondence to $T^*F \times B$. This sits inside $T^*(F \times \mathbb{R}^D)$. The objects will live here, and you have the double $T^*((F \times \mathbb{R}^D)^2)$, and so we get sheaves on $(F \times \mathbb{R}^D)^2 \times \mathbb{R}$. This will be a monoidal category that acts on $T^*(F \times \mathbb{R}^D)$. Sheaves on $(X \times X)$ always looks like matrices and acts on sheaves on X.

So you have M and then you have M_{cl} containing an algebra A, and you want to lift it to M. This should be governed by a 3-category. You need some kind of, it's not an algebra but a Batalin-Vilkovisky algebra. So the problem should have a canonical solution (not unique). Then after you lift this algebra you have a solution to the problem.

If you just try to lift it in a more straightforward way, you can construct a lower level deformation theory, but this doesn't lead to a solution. You can construct a deformation theory but you don't know how to solve it. It's, you don't have control. It just goes beyond any control.

Should I now become a little bit more concrete? Since I want to do everything on the dga level, first I need a dg category of sheaves on X, as opposed to just the derived category. You get an Abelian group of homs in the derived category but we'll have a complex. So this looks slightly unusual but it will be convenient for me. This won't be sheaves but cosheaves. I'll assume that X is locally compact. Then for such a space, you have a well-defined notion of derived sections with compact support on an open set $R\Gamma_c(U, \mathscr{F})$. Let me call this $\mathscr{F}_c(U)$. These have a different covariance. If $U_1 \subset U_2$, you can extend sections by 0, so $\mathscr{F}_c(U_1) \to \mathscr{F}_c(U_2)$. So this is a functor $\mathscr{F}_c : Open \to Ab^\bullet$ where $U_1 \to U_2$ if $U_1 \subset U_2$. So this is a precosheaf or a copresheaf.

The condition to make such a thing into a cosheaf is the following. If you have an open covering $\cup U_a = U$, you can write

$$\to \bigoplus \mathscr{F}(U_a \cap U_b) \to \bigoplus \mathscr{F}(U_a) \to \mathscr{F}_c(U).$$

This is the Chech complex. So $\mathscr{F}$ is called a cosheaf if all Chech complexes are acyclic.

So $CoSh_X \subset Fun(Open, AB)$ but I want to restrict to cofibrant functors. Many cofibrant objects can be described as filtered things $F^0\mathscr{F} \subset F^1\mathscr{F} \subset \cdots$ where $F^i\mathscr{F}/F^{i-1}\mathscr{F}$ are free. The differentials will all go down in the filtration. Then any object in the larger category of functors can be replaced with cofibrant functors. This is our real category of cosheaves. This is equivalent to a more usual category of derived sheaves. So for instance, take complexes of injective sheaves. This will work for manifolds, and you have a functor from this to the first model. [Functor went by too fast.]

You have a functor from this first one to the contravariant version. If you have a compact subset, and a cosheaf, define $\mathscr{F}(K)$ to be the cone of $\mathscr{F}_c(X\backslash X) \to \mathscr{F}_c(X)$. This will behave as a sheaf, it will have a restriction map. Then define $\mathscr{F}_U$ to be the derived inverse limit of $\mathscr{F}(K)$ for $K \subset U$. You can also define the sheaf in the opposite direction.

Then let me talk about, that's the last thing I want to do today is say what the microsupport is. Rather I will do it in a slightly different language. Let me fix $\Omega \subset T^*X$, an open conic subset. I will define $\mathscr{F}$ to be non-singular on Ω, but what does that mean? Informally, it means the following. So first of all let us consider, well $\mathscr{F}$ is a sheaf or cosheaf on X. Consider a domain in X with a smooth boundary. At every point in the domain $x \in \delta U$ has an exterior normal n_x, which is a ray in $T^*_x X$. So for instance $f \leq 0$, use that to define our variety near this point. Then $nx = \mathbb{R}_{>\geq 0} df$. If you change your equation, the ray won't change.

Then we can ask whether our ray belongs or doesn't to Ω. If $n_x \in \Omega$ then we say that x is Ω-non-singular. If it's not, then x is Ω-singular. Then let us perturb the nonsingular part of the boundary to make the domain a little larger, $\tilde{U} \supset U$. The perturbations should stay nonsingular. Let us call this an Ω-nonsingular deformation of U. Then we say that $\mathscr{F}$ is nonsingular on Ω if, in my language, $\mathscr{F}_c(U) \to \mathscr{F}_c(\tilde{U})$ is a quasiisomorphism for any nonsingular deformation $\tilde{U} \supset U$. Equivalently, nontrivially, if the usual restriction map $\mathscr{F}(\tilde{U}) \to \mathscr{F}(U)$ is a quasiisomorphism.

Now let us consider some examples. Let $x \in \mathbb{R}$ and you have a skyscraper. If you take $T^*(\mathbb{R}\backslash 0)$ it will be nonsingular. You don't have a lot of possibilities. you have an upper ray or lower ray. The upper ray would mean that you can freely change the right edge of your segment, including 0. [Another example, too fast]

On the arXiv, you can look at Guillermou-Kachuwara-Schapira, around this year. Maybe there is Guillermou-Schapira. Maybe Tsygan has something too.