MICROLOCAL CATEGORIES
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Let us try to get as far as possible. Let me remind you about last time. We need to come up with a model of our dg category of sheaves. Our choice was the following one. We worked not with sheaves but cosheaves on X, which is a full subcategory of the category of pre-cosheaves on X which is the category of functors from open sets in X to Abelian groups. If $U \subset V$ then you get $F(U) \to F(V)$. For a complex of injective sheaves with X a smooth manifold, i.e., locally compact and of finite injective dimension, then to a sheaf F you can associate a cosheaf $F_c(U) = \Gamma_c(U, F)$, sections with compact support. Then we also defined the notion of, it's not a big deal for many questions, but sometimes it's import to have this nice model. You need to work not with all functors, but with "cofibrant" functors. It's a technical condition and maybe we will skip it, functors which admit filtrations so that the associated graded are free (these are representable functors). There is a more conceptual definition, but let me just say that.

We talked about the singular support on a manifold. We defined the notion of a sheaf being nonsingular. So if $\Omega \subset T^*X$ was a conic open subset, and we defined what it means for F to be Ω-nonsingular. We considered Ω-nonsingular deformations of open sets. At each point you have an exterior normal n_x which is a ray in T^*_xX. So we say that x is Ω-nonsingular in δU if $n_x \in \Omega$. A nonsingular deformation, we are allowed to deform the nonsingular part of the boundary, so $U' \supset U$ is an Ω-nonsingular deformation if U' can be obtained from U by perturbing the Ω-nonsingular part of the boundary in an Ω-nonsingular way. We say that a cosheaf F is Ω-nonsingular if for any nonsingular deformation, the induced corestriction map $F(U) \to F(U')$ is a quasiisomorphism. For a usual sheaf you should use the restriction map. These are equivalent but that's nontrivial.

Now we need to add some detail without complete definition. We haven't seen the zero section so far. Now how to say that something is Ω-nonsingular if Ω contains a zero section. So without loss of generality, say $\Omega = T^*\mathring{X}$ where $\mathring{X} \subset X$ is open. Here we'll say whenever $U \subset \mathring{X}$, the deformation $\emptyset \to U$ is Ω-nonsingular. I forgot to impose the axiom when I defined my sheaf that $F(\emptyset) = 0$. This implies that my sheaf is 0 at such a U. To say that F is nonsingular when Ω contains a zero section, that's the same as saying F is zero.

You have many open subsets and you have to check separately if they are singular or not. It turns out that being nonsingular on several subsets, then it is nonsingular on their union as well. So if F is Ω_a nonsingular, it is $\cup\Omega_a$-nonsingular.

You need to geometrically, if you have a deformation which is nonsingular with respect to this union, you need to present it as a composition of deformations which are nonsingular with respect to each one. This is not trivial but is true, even when this is infinite.

Therefore there exists the largest subset Ω where F is nonsingular. The complement is a closed subset, and that's the singular support, so that's $T^*X\backslash\Omega$. So if F is 0 then the singular support will be empty; if F is locally constant it will be the zero section. If I have a domain with smooth boundary, I can consider a constant sheaf on U. The singular support of this sheaf over x, $SS(A_U)_x$ is

$$\begin{cases} x \notin \bar{U} & SS(A_U)_x = \emptyset \\ x \in U & SS(A_U)_x \text{ is the zero section} \\ x \in \delta U & SS(A_U)_x = \bar{n}_x \end{cases}$$

If you take A on the closure of such, well, $SS(A_{\bar{U}})_x = -SS(A_U)$, because of Verdier duality.

Then I need to do one more thing. We have operations on sheaves and then we'll see how the singular support behaves. We won't use all six functors but only half of them. We need to define functors $csh(X) \to csh(Y)$. We'll define them by $csh(X \times Y)$. so we want to define maps

$$csh(X) \circ csh(X \times Y) \to csh(Y)$$

These will be at least functors of pullback and pushforward, f^{-1} and $f_!$, not the tensor product, but certainly these two. This will encode all functors we need. How will we define it. It's technical, how to make it in the simplest way in our framework. If you look at sheaves or cosheaves on a topological space, you can look on a basis, so we can look at product open subsets. We will, instead of $csh(X \times Y)$, we will define $cpsh(X \cdot Y) : Open(X) \times Open(Y) \to Ab$. So we have cosheaves here, and cofibrant objects. That's what we're going to do. Then we'll have $csh(X) \subset Fun(Open(X), Ab)$ and $csh(X \cdot Y) \subset Fun(Open(X) \times Open(Y), Ab)$. By the laws of tensor algebra, you need to find a functor, a pairing between $Open(X)$ and itself. So we need to define something $K_{XX} \in Fun(Open(X) \times Open(X)^{op} \to Ab)$ So most natural is to take the diagonal Δ and write something that approximates the constant sheaf. So $\mathbb{Z}_\Delta \mapsto \tilde{\mathbb{Z}}_\Delta$, a flabby resolution, and define $K_{XX}(U \times V) = \tilde{\mathbb{Z}}_\Delta(U \cap V)$. Then we can define a resolution, well, if I have $F : Open(X) \to Ab$ and $G : Open(X) \times Open(Y) \to Ab$, then I can define $K \otimes (F \boxtimes G)$ and define that as $F \circ G : Open(Y) \to Ab$. Then the result doesn't depend on choice of resolution, up to quasiisomorphism. Then this agrees with the convolution of sheaves, which, well, you have $X \times Y \to X \times X \times Y$ and you have projections onto X, $X \times Y$, and Y. Then $F \in D(X)$, $G \in D(X \times Y)$, then their convolution is defined as $F \circ G = \Delta^{-1}(p_1^{-1} F \otimes p_{23}^{-1} G)$.

So let me make some motivation. Let us try to estimate the singular support on $F \circ G$. Let us switch to closed sets, and sections on compact sets, this the honest cone of sections with compact support on $X \backslash K$ to $F(X)$.

If I let K be a compact subset of Y, then $F \circ G(K)$ is $(F \boxtimes G)(\Delta_X \times K)$. If F is a sheaf on X, then G belongs to $sh(X \times Y)$, and I can define their exterior product $\boxtimes$ which belongs to $D(X \times X \times Y)$. You get something only defined on $X \cdot X \times Y$, but $Open(X) \times Open(X \times Y) \subset (X \times X \times Y)$, and you define this by induction (or Kan extension). The singular support is the product.

Then the rest is clear, I can multiply by the diagonal, it's skewed, and the claim is that sections of the convolution on a compact set, if X is noncompact it should be sections with compact support. Now K can be the closure of our thing with closed boundary. So now we can look at $\Delta_X \times K$. This is not a domain with smooth boundary because it has the diagonal in it.

Let me be informal. I'll make a mistake but correct it. It looks like a closed submanifold. At every point you'd want to have one and only one normal. It's probably the limit of small tubes, which gives you many normals. If $Z = \Delta_X \times K$, if $Z \subset X$ is a closed submanifold, then n_z, the forms that vanish on Z, is $(T^*_Z X)_z$. What should be in the singluar support of this set? You take a normal, put it in K, and multiply it into Δ_X. It will be singular if its image is singular.

Let us try to write it out. Now we have the special diagonal X in $X \times X$. Then the conormal $T^*_X(X \times X)$ is $T^*X \hookrightarrow T^*X \times T^*X$ via the identity times the antipode. Then I have (x, ω), where x is in the boundary of K and ω is normal to K. I want to ask whether $T^*_X(X \times X)x \cdot \omega \cap \cap(SSF \times SSG)$. We can derive the following wrong theorem: Let $(y, \omega) \in T^*X$, well, (y, ω) is in the singular support of $F \circ G$ if and only if there is a point $p \in T^*_X(X \times X)$ so that $p \times \omega \in SSF \times SSG$. This is not quite true because we've looked at tubular neighborhoods. Let me give the weakest version of this. You need to rewrite all this story as $T^*X \times T^*X \times T^*Y$, which holds a product of my singular support $SSF \times SSG$. Inside of this there is $T^*_X(X \times X) \times T^*Y$ which maps to T^*Y. Call the maps i and p. The theorem is correct if $pi^{-1}(SSF, SSG)$ is proper.

This convolution can be, first you restrict to the diagonal, then you restrict on Y.