MICROLOCAL CATEGORIES
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Last time we did the Fourier transform, now let me do a slight generalization. Imagine that our symplectomorphism $\phi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is close to the identity in some way that I won't specify. Then you can write down something like $(\tilde{q},\tilde{p}) = \phi((q,p))$. Let us be more precise: assume $(\tilde{q},p)$ form a coordinate system in $T^*\mathbb{R}^n$. If you look at the graph of ϕ, this condition is equivalent to saying that the graph of ϕ projects nicely by this pair of coordinates. In other words, what you can do, if you just look at $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, the coordinates here are $\{(q,p,\tilde{q},\tilde{p})$, and we can apply the Fourier transform to the second pair of variables, so $\mathcal{F}_{(\tilde{q},\tilde{p})}$ takes this to $(q,p,-\tilde{p},\tilde{q})$, and then if I look at the graph $\Gamma_\phi C$, this goes by this symplectomorphism to $\mathcal{F}(\Gamma_\phi)$ which will also be a Lagrangian.

Then there is a function $S(q,\tilde{p})$ depending on these variables so that $-p = \frac{\partial S}{\partial q}$ and $\tilde{q} = -\frac{\partial S}{\partial \tilde{p}}$. The identity corresponds to the function $S = -q\tilde{p}$. Then we will get $p = \tilde{p}$ and $q = \tilde{q}$. So it's convenient to write $S = -q\tilde{p} + S_0$. Symplectic geometers would kill me for such an exposition, but this is how I first learned it. With this knowledge at hand we can write down a quantization formula. I should write something sitting on the Lagrangian $\mathcal{F}(\Gamma_\phi)$ and then apply the inverse of the Fourier transform. To write something that lies on this Lagrangian, you just say $\mathbb{Z}_{\{t+S_{(q,\tilde{p})}-q\tilde{p}\geq 0\}} * \underbrace{\mathbb{Z}_{t+\tilde{p}\tilde{q}\geq 0}}_{\mathbb{F}^{-1}}$.

Let me write

$$R\pi_!\mathbb{Z}_{t+S_0(q,\tilde{p})+\tilde{p}(\tilde{q}-q)\geq 0}$$

where $\pi : \mathbb{R}^n_q \times \mathbb{R}^n_{\tilde{q}} \times \mathbb{R}_p \to \mathbb{R}^n_q \times \mathbb{R}^n_{\tilde{q}}$.

So to give a more down to earth explanation. You have a Morse theory that says that your functor can be calculated by looking at critical points of your function, and there will be only one critical point, $\tilde{p} = f(q,\tilde{q})$, maybe I'm wrong, I guess it may not have a unique projection. If the projection onto the q-$\tilde{q}$ plane is bad there will be more critical ponits. But if there is only one, then this $f(q,\tilde{q})$ gives you [unintelligible]. For the identity map, you will have no critical points at all or a critical point at 0 so it's a δ function. So that's how you quantize it.

The Hamiltonian symplectomorphism can always be quantized. Let us do an interesting example. Let us try to quantize a linear symplectomorphism. So you have $Sp(2D)$ acts on $T^*\mathbb{R}^D$. Then from what follows here you can immediately quantize all transformations close enough to the identity. Let me do it as an introduction. So you have a neighborhood of the identity that can be quantized in this manner. But any element of $Sp(2D)$ can be written as a product of elements close to the identity. The resultant answer will depend on how you write your group element as a product of elements close to the identity. So you are prescribing a path which is a broken geodesic, and the answer will depend on this path, since $Sp(2D)$ is not simply connected. You can pass to the universal cover $\widetilde{SP}(2D)$ and hope that this resolves the ambiguity, and this is in fact the case.

$\widetilde{Sp}(2D)$ acts on $T^*\mathbb{R}^D$ should mean you have a sheaf like this: $D_{>0}(\widetilde{Sp}(2D)\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R})$. Let me call $\widetilde{Sp}(2D)$ by G, then we know we have the action A of G on $T^*\mathbb{R}^D$ by symplectomorphism, and we have the form $\alpha = pdq$ on $T^*\mathbb{R}^D$, and if I pull back by A^*, on the second factor I'll get something homologous to α on the second factor, so $p_2^*\alpha + dH + \theta$ where θ is 0 along $T^*\mathbb{R}^D$. It's obvious that you can do this at each point of G separately. It's not obvious you can do it globally, but you can find H rather explicitly. Then θ will be a one-form with values in T^*G, it can be written alternatively as a map $G\times T^*\mathbb{R}^D \to T^*G\times T^*\mathbb{R}^D$ which is a section.

Now if I pull back, it follows that the difference between $A^*\alpha_{T^*\mathbb{R}^D}$ and $\theta^*\alpha_{T^*(G\times\mathbb{R}^D)}$ it's a closed form, in fact dH, so we have constructed a Lagrangian correspondence.

So $Im(-A)\times\theta\subset T^*G\times T^*\mathbb{R}^D\times T^*\mathbb{R}^D$ is a Lagrangian manifold. We can look for an object on the base whose singular support is precisely this one. To make it even more rigid, we can do it further, I can take the cone of this, which will be in the product. Since I have the function H, I have $Im\{(-A)\times\theta\times H\}$ which will be Legendrian, in $T^*(\underbrace{G\times\mathbb{R}^D\times\mathbb{R}^D}_{X})\times\mathbb{R}$. Let me call this manifold Λ, and I want to take the cone of Λ, which will be points (X,ω,t,k) with $k>0$ and $(x,\frac{\omega}{k},t)\in\Lambda$ where $x\in X$ and $\omega\in T_x^*X$, $(t,k)\in T^*\mathbb{R}$. Instead of quantizing a Lagrangian manifold, which would let t be completely free, we fix t so that it is completely rigid. It will be good to make our conditions as rigid as possible.

Now we can simply consider the category of all objects, C_Λ, in $D_{<0}(G\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R})$ consistitng of all $\mathcal{F}$ so that $SS\mathcal{F}$ is in $Cone\Lambda$, where singular support means as usual the intersection with the upper half-space.

Let $U\subset G$ be open, then we can construct C_Λ^U which is in $D_{<0}(U\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R})$ which is defined the same way, that the singular support of $\mathcal{F}$ is contained in $\Lambda\cap T^*[unintelligible]$.

Claim 1. *If U is contractible and has small enough diameter, we have an equivalence $C_\Lambda^U\cong Ab^{\cdot}$ (where small enough means $U=gU_0$ where U_0 is a small neighborhood of the identity where I can apply my generating function procedure)*

I can choose any quantization and find an object in this category in this way, and then I can multiply by any complex, then I can show there are no other objects.

Now we need to prove that for each nice U which is a small enough neighborhood, you can make an object and you'd like to glue them together.

How can you do it? There is a general theorem on this. If your G is 3-connected, then you can always find a global object of this category. You can find a global object so that you get a 1-dimensional Abelian group up to some shift. Let me comment on how you can prove it. Basically I don't want to pronounce these terrifying words about sheaves of categories, let me jest give a construction of an object on the whole category. Use a good covering $\cup U_a$ where each one and all intersections are contractible, like geodesically convex neighborhoods. Let me consider all possible intersections, $\{U_i\}$, nonempty, I would say, and they will form a partially ordered set by inclusion, where the $i\in\mathscr{I}$—$\mathscr{I}$ is a poset. Then for each i there is $\mathcal{F}_i$ in $\mathscr{C}_\Lambda^{U_i}$. Take $j_i: U_i\to G$, and then we can look at $j_{i!}\mathcal{F}_i\in D_{>0}(G\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R})$ and we can look at $Rhom(j_{i!}\mathcal{F}_i, j_{i'}\mathcal{F}_{i'}) = R^{\cdot}hom|\mathcal{F}_i,\mathcal{F}_{i'}|_{U_i}\cong\mathbb{Z}[d_{ii'}]$ and there is a cocycle condition, for $i<i'<i''$

we ha $d_{ii'} + d_{i'i''} = d_{ii''}$ and then there is an e_i so that $d_{ii'} = e_{i'} - e_i$ so $\mathcal{F}_i \to \mathcal{F}_i[e_i]$. [Missed some]

Now what can you do? Since all of our homs $hom(j_{i!}\mathcal{F}_i, j_{i'!}\mathcal{F}_{i'}) \cong \mathbb{Z}$, we can look at $\tau_{\leq 0}$, I can look at this as the morphisms of a new category with objects $\mathscr{I}$, call this category Ψ, then this maps to Ψ_0 which has the same objects but homs H^0 (This can be made into a complex). This will be a weak equivalence of categories. The Ψ_0 will be a basic category. You can get into trouble here with $\pm$. We have an ambiguity with generator of $\mathbb{Z}$. Then $g_{ii'} \in hom(j_{i!}\mathcal{F}_i, j_{i'!}\mathcal{F}_{i'})$, and it's natural to want to compare $g_{i'i''} \circ g_{ii'} = g_{i''i}$ up to the sign $(-1)^{d_{ii'i''}}$ where $[d_{ii'i''}] \in H^2(G, \mathbb{Z}/2\mathbb{Z})$ and for us this is 0.

By changing generators you can eliminate this sign, and so then Ψ_0 is equivalent to $\mathbb{Z}[\mathscr{I}]$. From the beginning we constructed a functor $\mathcal{F} : \Psi$ to $D_{>0}(G \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R})$ so you have an A_∞ functor from $\mathbb{Z}[\mathscr{I}]$ to the cosheaf category, you take a hocolim, and if you don't have this knolwedge you will spoil your sheaf and the result will be unpredictable.

Then you have $S := \mathrm{hocolim}_\Psi \mathcal{F}$. Then you can say all objects $S \in \mathscr{C}_\Lambda(G)$ which have [unintelligible].

I'm running out of time, so next time let's say what happens when we take composition of two objects in G. Next time let's talk about how the center of the metaplectic group acts. There is another thing, we will try to explain, this G contains as a subgroup $GL(D)$, the coordinate changes, which lifts to this universal cover. We will see what happens when we lift this, this will be the source of the Fukaya category data. You will need [unintelligible]grading (because of the center) and a spin structure (which comes from the GL action).