MICROLOCAL CATEGORIES
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So we are working with cosheaves on Y, a full subcategory of functors from open sets of Y to Abelian groups, imposing the condition that the Cech complex should be isomorphic to [unintelligible]. All fibrations are surjective, and every object has a cofibrant representative, so we can use cofibrants. If you have $X\times\mathbb{R}$, you have $D_{>0}(X\times\mathbb{R})$ which is $D(X\times\mathbb{R})/D_{T^*_{\leq 0}(X\times\mathbb{R})}(X\times\mathbb{R})$. This can [unintelligible]the cosheaves so that $\mathcal{F}(U\times(-\infty,a))\sim 0$. So we restrict further to look at $J=\{(a,\infty)\}$ and then $\mathscr{A}(X)$ are functors from $Open(X)\times J$ to Ab, and you look at $D_{>0}(X\times\mathbb{R})$ and so if we impose some conditions, well, we can restrict to $\mathscr{A}(X)$, we want this to be an equivalence of categories. So there should be conditions. It's convenient for $a\in\mathbb{R}$ to have $\mathcal{F}_{(a,\infty)}$ be a copresheaf on X, and in terms of this we impose the following conditions: $\mathcal{F}\in\mathscr{A}(X)$ if and only if

(1) $\mathcal{F}_{(a,\infty)}$ is a cosheaf on X,
(2) the homotopy colimit for $b>a$ of $\mathcal{F}_{(b,\infty)}$ is equivalent to $\mathcal{F}_{(a,\infty)}$, a quasiisomorphism. To do this, you could say that for each open set the colimit of $\mathcal{F}_{(b,\infty)}(U)$ is $\mathcal{F}_{(a,\infty)}(U)$, and
(3) $\mathcal{F}_{(-\infty,\infty)}\sim 0$.

Then $\mathscr{A}(X)$ are cosheafs filtered by the reals subject to a continuity axiom and a vanishing axiom.

It's convenient to have both pictures. Let's talk about microsupport.

If $\mathcal{F}\in D_{>0}(X\times\mathbb{R})$ then the singular support $SS\mathcal{F}$ can be thought of or defined as something in $T^*_{>0}(X\times\mathbb{R})$, it's the intersection with the usual singular support. It turns out that in some good situation you have something on the zero section, but nothing below.

We mostly allow, our subsets are cones of subsets in T^*X, so we say that the microsupport $MS(\mathcal{F})$ is a subset of T^*X, and we'll write $MS(\mathcal{F})\subset C$ if and only if $SS(\mathcal{F})$ is in the cone of C.

We can try to look at some simple cases. Given a closed subset C we can ask what sheaves have support inside it. The simplest case is when C is the zero section of T^*X. You can prove the following statement: Let $\mathcal{F}\in\mathscr{A}(X)$. Then the microsupport of $\mathcal{F}$ lies in the zero section T^*_XX if and only if $\mathcal{F}_{(a,\infty)}$ are locally constant cosheaves on X.

Then if $X=\mathbb{R}^D$, we have an equivalence, we can construct a functor $\pi:\mathscr{A}(X)\to\mathscr{A}(pt)$ so that $\pi\mathcal{F}(a,\infty)=\mathcal{F}(X\times(a,\infty))$. You should restrict π to all objects whose microsupport is the zero section, this version of π is an equivalence.

This is because the category of locally constant sheaves on $\mathbb{R}^d$ is equivalent to sheaves on a point.

That's the simplest example. Then we can work out the case when our Lagrangian is the graph of the differential of some smooth function. Then you can consider a diffeomorphism $X\times\mathbb{R}\to X\times\mathbb{R}$ which sends $x,t\xrightarrow{\phi}x,t+f(x)$. It's harder to check that $D(X\times\mathbb{R})\to D(X\times\mathbb{R})$ by the induced functor ϕ_*, and it preserves the category $D_{>0}(X\times\mathbb{R})$, and another claim is that if I start with a sheaf supported on L, then ϕ_* of the sheaf will be supported on the zero section. This trick is hard to perform in $\mathscr{A}(X)$, but here it's easier.

Because of that, the category of all sheaves microsupported on L, if X is contractible, then your category is identified: $D_{>0}(X\times\mathbb{R})_L\to A(pt)$ and this is the same as $\mathscr{A}(X)_L$.

I forgot to do a more fundamental thing, define the action of the Novikov ring on all this business.

An action on the Novikov ring, you have a functor of sheaves $T_c:\mathscr{A}(X)\to\mathscr{A}(X)$ along the real axis: $T_c\mathcal{F}(U\times(a,\infty))=\mathcal{F}(U\times(a-c,\infty))$. You have a canonical map $Id\to T_c$ for $c\geq 0$ and by the same reason $T_c\to T_d$ for $d\geq c$. I'll introduce a Novikov ring, but let me write this first.

$\Lambda^{\mathbb{R}}$ is a graded ring graded by the real numbers, and $\Lambda^{\mathbb{R}}_c=\mathbb{Z}e^{-\frac{c}{\hbar}}$ where this is just a formal basis element, with multiplication $e^{-\frac{c}{\hbar}}e^{-\frac{d}{\hbar}}=e^{-\frac{c+d}{\hbar}}$, and we say that for $c<0$ that $\Lambda^{\mathbb{R}}_c=0$.

We'll look at $\Lambda^{\mathbb{R}}$-mod. We need to add something. The recipe will be very simple. We want to set $\underline{hom}_c(\mathcal{F},\mathscr{G})=hom(\mathcal{F},T_c\mathscr{G})$. It's nice if we start by first choosing a cofibrant replacement for everything. That's basically it because for $d\geq c$ we get induced maps $\tau_{cd}:\underline{hom}_c(\mathcal{F},\mathscr{G})\to\underline{hom}_d(\mathcal{F},\mathscr{G})$, and this is the structure of such a module, this is the action of $e^{-\frac{d-c}{\hbar}}$.

This is enriched, so $\mathscr{A}(pt)$ is basically just a module over this Novikov ring. So if $\mathcal{F}_{-a}=\mathcal{F}(a,\infty)$, this gives us a map from $\mathscr{A}(pt)$ to $\Lambda^{\mathbb{R}}$-modules. We'll get only some of these because of the continuity axiom.

Maybe now I will postpone the abstract nonsense. In principle we need to define $\mathscr{A}(X\cdot Y):Open(X)\times Open(Y)\times\mathcal{J}\to Ab$, and you have a convolution functor $\mathscr{A}(X\cdot Y)\times\mathscr{A}(Y\cdot Z)\to\mathscr{A}(X\cdot Z)$. I will skip this generalization, and let me instead quantize symplectomorphisms of T^*X. Suppose I have a symplectomorphism Φ on T^*X. We can also look at $\hat{\Phi}$ on $\mathscr{A}(X)$. We can say that $\hat{\Phi}$ corresponds to Φ if for all $\mathcal{F}\in\mathscr{A}(X)$ the microsupport of $\hat{\Phi}(\mathcal{F})$ is Φ of the microsupport of $\mathcal{F}$.

There is a recipe to do this. Take the graph $\Gamma_\Phi=(a,x;\Phi(x))$ in $T^*X\times T^*X$. [unintelligible]choose $\hat{\Phi}\in\mathscr{A}(X\cdot X)$ and this will do the job, $MS(\hat{\Phi})\subset\Gamma_\Phi$.

Let's do the example of a Fourier transform. So $\mathbb{F}$ acts on $T^*\mathbb{R}^D$, which can be identified with $\mathbb{R}^D\times\mathbb{R}^D$, and $\mathbb{F}(q,p)=(-p,q)$. Then $\Gamma_\Phi=(q,p,-p,q)$. we should say $T^*(\mathbb{R}^D\times\mathbb{R}^D)$ is (q_1,p_1,q_2,p_2), and you can see that the porjection on the first and third coordinates is one to one. Then Γ_Φ is [unintelligible].

So what you need to do is take a sheaf supported on Γ_Φ. An example is the constant sheaf on the set $(q_1,q_2,t):t-q_1q_2>0$. Then if I have an object, let me give a hint of the convolution, let us call this object $\mathscr{F}$. Then $\hat{\mathbb{F}}$, I want to define, on $D_{>0}(\mathbb{R}^D\times\mathbb{R})$. If I take an object S there, I want to locate my sheafs on

$$\begin{tikzcd}
\mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R} \times \mathbb{R} \arrow[r, "p_{1,3}"] \arrow[d, "p_{1,2.4}"'] \arrow[dr, "a"] & \mathbb{R}^D \times rR \\
\mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R} & \mathbb{R}^D \times \mathbb{R}
\end{tikzcd}$$

where $a(q_1, q_2, t_1, t_2) = (q_2, t_1 + t_2)$. Then $\hat{\mathbb{F}}(S)$ is $Ra_!(p_{1,3}^{-1}S \otimes p_{1,2,4}\mathscr{F})$. This will change microsupport in the prescribed manner, so $\hat{\mathbb{F}}\delta_0 \boxtimes \mathbb{Z}_{t\geq 0}$ is $\mathbb{Z}_{\{(x,t)|t\geq 0\}}$. it is interesting to ask, what is $\mathbb{F}$ to the fourth power. Actually, $\mathbb{F}^2$ will be reflection, if you do it with functors, it's not going to be—

[Kevin: metaplectic.]

You need to write the convolution of the two kernels, it will be something like $t-(q_1-q_3)q_2 \geq 0$, and you project [unintelligible].

[Mohammed: I'd like to see that.]

Instead of doing what is written here, you can first do $\mathbb{Z}_{\{t-xq_2\geq 0\}}$. I claim that $Rp_!$ of this will be a skyscraper. For each x I should look at the stalk at x. Formally we have $\mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}$, restrict to $x_0 \times \mathbb{R}$ and we project to $\mathbb{R}^D \times \mathbb{R}$. Since lower shriek commutes with restriction, [unintelligible]. If $x_0 \neq 0$, then we're looking at $t_0 - xq \geq 0$, and this is a closed ray. The cohomology with compact support is 0. Outside of 0 it will be 0. With 0 this business will disappear. We get $\mathbb{Z}_{t\geq 0} \times \mathbb{Z}_{\mathbb{R}^D}$. Now we take $Rp_!$ and we will get the shift by D units. What I said is true for one.

I think we have to stop now. The pretalk is at 1, right? I think it would be better to stop here anyway.