MICROLOCAL CATEGORIES
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Let me start with something very simple. I defined the convolution $\circ$ as an operation

$$csh(X) \times csh(X \cdot Y) \to csh(Y)$$

but there is a slight generalization:

$$csh(X \cdot Y) \times csh(Y \cdot Z) \to csh(X \cdot Z)$$

where $X \cdot Y$ is $Open(X) \times Open(Y)$. So if I have functors $\mathscr{F} : Open(X) \times Open(Y) \to Ab$ and $\mathscr{G} : Open(Y) \times Open(Z) \to Ab$ and then $K^{YY} : Open(Y)^{op} \times Open(Y)^{op} \to Ab$, so then $\mathscr{F} \circ \mathscr{G} := \mathscr{F} \boxtimes \mathscr{G} \otimes_{Open(Y) \times Open(Y)} K^{YY}$.

This is a slight generalization from what we did last time. Now I can start to do things.

So basically, you can estimate singular support. So you have $SS\mathscr{F} \times SS\mathscr{G} \subset T^*X \times T^*Y \times T^*Y \times T^*Z$ and you have a conormal bundle to the diagonal and you get a Lagrangian correspondence: You can pull back along i to $T^*X \times T^*_Y(Y \times Y) \times T^*Z$ and push forward along p to $T^*X \times T^*Z$. So if p is proper on $i^{-1}(SS\mathscr{F} \times SS\mathscr{G})$ then $SS(\mathscr{F} \circ \mathscr{G}) \subset pi^{-1}(SS\mathscr{F} \times SS\mathscr{G})$.

If I want to mimic the Fukaya category, I want my sheaves to be something like $D(X \times \mathbb{R})$. So first of all, why we should do it, let me remind you, if I have $A \subset T^*X$ then I can define its cone $Cone(A) \subset T^*X \times T^*\mathbb{R}$. This $Cone(A)$ consists of tuples (x, ω, t, κ), where $x \in X, \omega \in T^*_x X, t \in \mathbb{R}$, and $\kappa \in T^*_t\mathbb{R}$. Then $Cone(A)$ consists of points so that $\kappa > 0$ and $(x, \frac{\omega}{\kappa}) \in A\}$ This is contained within $T^*_{>0}(X \times \mathbb{R}) = (x, t, \omega, \kappa) | \kappa > 0\}$. We can also look at $T^*_{\leq 0}(X \times \mathbb{R})$. Now we can look at, if A is a closed subset of T^*Y then I will call $csh_A(Y)$ will be the full subcategory of $\mathscr{F} \in csh(Y)$ with singular support in A. We can consider $csh_{T^*_{\leq 0}(X \times \mathbb{R})}(X \times \mathbb{R})$, and we want to quotient $csh(X \times \mathbb{R})$ by this category, and call it $\mathscr{A}(X)$. This is extremely nonconstructive and we'd like to simplify it.

Let us again look only at open sets which are products. Look at $csh(X \cdot \mathbb{R}) \leftarrow csh(X \times \mathbb{R})$, and this restriction is a weak equivalence of categories. I don't want to look at all possible open subsets, but only at rays, and consider the subcategory in $Open(\mathbb{R})$ which consists of $\{(-\infty, a); (a; \infty), (-\infty, \infty)\}$ We can restrict there and still formulate the sheaf condition on this subcategory. Call this $Rays$. So you have $Open(X) \times Rays \to Open(X \times \mathbb{R})$ so $csh(X \times \mathbb{R}) \to Funct(Open(X) \times Rays, Ab)$. We want to impose a sheaf condition which makes a presheaf into a sheaf. We want a full subcategory $C(X)$ which consists of functors satisfying the sheaf condition. We need some notation to define it. If I have a functor $\mathscr{F} : Open(X) \times Rays \to Ab$, then $\mathscr{F}_{(a,\infty)}$ is a functor $Open(X) \to Ab$, with $\mathscr{F}_{(a,\infty)}(U) = \mathscr{F}(U \times (a, \infty))$. Then include into $C(X)$ all sheaves $\mathscr{F}$ satisfying $\mathscr{F}_{(a,\infty)}$, $\mathscr{F}_{(-\infty,a)}$, and $\mathscr{F}_{(-\infty,\infty)}$ are cosheaves on X. Secondly, I can just write it down:

$$\lim_{a_1 > a_2} \mathscr{F}_{(a_1,\infty)}(U) \xrightarrow{\sim} \mathscr{F}_{(a_2,\infty)}(U)$$

So you need the direct limit to be $\mathscr{F}_2$. If we allow a_2 to be negative ∞, you need the opposite condition. Then the claim is, this will be an equivalence of categories.

So you'll get two families of cosheaves, continuous on the real line. Now in this language it's easy to say what it means for one of these to be supported on the lower half-plane.

Claim 1.

$$SS\mathscr{F} \subset T^*_{\leq 0}(X \times \mathbb{R}) \iff \mathscr{F}_{(-\infty,a)}(U) \xrightarrow{\sim} \mathscr{F}_{(-\infty,\infty)}(U)$$

for all U.

Then $C(X) \cap csh_{T^*_{\leq 0}(X\times\mathbb{R})}(X \times \mathbb{R}) \overset{I}{\hookrightarrow} csh(X \times R)$ has a left adjoint L. So $L\mathscr{F}_{(a,\infty)} = \mathscr{F}_{(a,\infty)}$ and you can define $L\mathscr{F}_{(-\infty,a)} = \mathscr{F}_{(-\infty,\infty)}$.