MICROLOCAL CATEGORIES
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Last time, I'm tired of writing $\mathbb{R}^n$, and so I'll call it E, and $G = \widetilde{Sp}(2n)$, which you should think of as $E \oplus E^*$. Then we constructed (not quite explicitly) an object $S \in D_{>0}(G \times E \times E \times \mathbb{R})$. By some general nonsense you can extend something uniquely that is explecitly defined only in a neighborhood. The first thing I'd like to do today is finish the construction. I can make S_g for $g \in G$, and that's a quantization of the symplectomorphism $g : E \oplus E^*$ or T^*E to itself. You can consider the graph of g which is a Lagrangian (well, you should reflect in the first argument). A quantization means that the microsupport of S_g is in this graph. If you want to do this in families over G there is a unique choice. Now we'd like to compare $S_g \circ S_h$ with S_{gh}. You should impose the condition $S_e = \mathbb{Z}_\Delta \boxtimes \mathbb{Z}_{\geq 0}[\dim G]$. Last time I think I left off the shift. Well, actually, oh, probably, anyway, we will get to it later. Let's just define it with the shift. It's something that should be a continuous family. So induce a monoidal structure on the category $D_{>0}(G \times E \times E \times \mathbb{R})$, morally this category is sheaves on G and sheaves on $E \times E \times \mathbb{R}$. You have separate convolutions on each of these, and so we should define this here then on the differential graded level. I don't know how to do it in a straightforward way. So you already have a big problem, if you want to define the convolution product on G, let us define convolution of sheaves on G to make it a monoidal category.

Normally convolution is defined as follows: you have multiplication $G \times G \to G$, and you define the convolution as a derived pullback $\mathcal{F} * \mathscr{G} = Rm_!(\mathcal{F} \boxtimes \mathscr{G})$. How do you define $m_!$, as a left adjoint to the pullback $m^{-1} : cosh(G) \to cosh(G \times G)$, wait, no, here for, well, we want a differential graded model that is associative. You can do it as follows. You define $m_!$, you have a formula that sections with compact support on $m_!\phi|_U$ is the same as sections with compact support of ϕ on $m^{-1}U$. A lot of things get messed up if you try to mimic directly, but you can fix things by relaxing it a bit. So we'll construct a relaxed monoidal category. Let me switch directly to $G \times E \times E$, we'll construct $\mathscr{A}(G \cdot E \cdot E)$, only on the product of open sets. I don't know, maybe it's "lax" monoidal category. There are many versions but I mean this: for each arity n you have a tensor product T_n and for any planar tree, you want to insert these one into another, and you can put, associate to each vertex a functor, and if you compose these you will get a new functor T_t which will go $\mathscr{C}^{N(t)} \to \mathscr{C}$ where $N(t)$ is the number of inputs of the tree t. Our trees can also have 0-valence vertices, and for any such tree you can look at the space $\mathcal{F}(t) = hom(T_{N(t)}, T_t)$, where if you have a small tree, imagine you have a micro tree, and at each vertex you have another tree. You have insertion, and you have insertion maps for this $\mathcal{F}(t)$. You have t for each vertex in the interior of t, we are given tree t_v so that $N(t_v)$ is the arity of v. From this you can create a new tree by insertion. Then $\mathcal{F}(t)$ describes multilinear operations, (these should be complexes of natural transformations), multilinear operations

$$\mathcal{F}(t) \otimes \bigotimes_{v \in V} \mathcal{F}(t_v) \to \mathcal{F}(t\{t_v\}_{v \in V})$$

Here your valence is labeled by a tree instead of labeling by a number, so it's some kind of higher operad, a t-operad. You have a trivial t-operad (a t-operad for any planar tree you have a complex and for every insertion you have a composition which is associative) and the trivial one is $\mathbb{Z}$ in every tree. We define a lax monoidal structure a functor $Rtriv \to \mathcal{F}$. You can impose a stronger restriction. You can additionally demand that a generator of $Rtriv$ should map to a weak equivalence.

Usually such thing cannot be constructed explicitly but you can prove that there is no obstruction. We will see that the space $t \leq 0$ of these homs $T_{N(t)}, T_t$ will be 1-dimensional. Our category was something like $\mathscr{A}(G \cdot E \cdot E)$. To represent these functors by kernels, we can define a convolution

$$\mathscr{A}(G \cdot E \cdot E)^n \circ \mathscr{A}(\underbrace{G \cdot E \cdot E}_{n+1 \text{ times}}) \stackrel{conv}{\to} \mathscr{A}(G \cdot E \cdot E).$$

Pick $\mathcal{K}_n$ in the second factor and then $T_n(\mathcal{F}_1, \cdots, \mathcal{F}_n) = conv(\mathcal{F}_1, \cdots, \mathcal{F}_n, \mathcal{K}_n)$ and we want $\mathcal{K}_n$ to be a representative of $\mathbb{Z}_{L_n}$ where

$$L_n \subset (G \cdot E \cdot E)^{n+1} = \{g_i, e_i, f_i\}_{i=1}^n = f_1 = e_2, f_2 = e_3, \cdots, e_i = e_{n+1}, f_n = f_{n+1}$$

(???)

It's easy to do [unintelligible]with the kernels.

Since usual convolution is strictly associative, it will turn out that $\mathcal{K}_t$ will be weakly equivalent to $\mathcal{K}_{n(t)}$. Then calculating $\mathcal{F}(t)$ will boil down to choosing, its cohomology will be $R^\bullet Hom(K_{n(t)}, K_{n(t)})$ which will be the same, we will have Euclidean spaces, and you'll be left with $H^*(G^\#)$, the cohomology of some power of G. If I now replace each of my spaces with their cutoffs $\tau_{\leq 0}\mathcal{F}(t)$ I'll get $\mathbb{Z}$. The induced composition will preserve the generators 1. I have a natural projection to the trivial operad which is a weak equivalence, and also a natural map to $\mathcal{F}$. So we are done. This analysis shows that if someone else tries to invent another version of a monoidal category it will be equivalent to this one.

After this we need to define, if we were in a usual monoidal category, we would want to endow our thing with an algebra structure. In this relaxed world, well, in the strict case we could consider the full operad $full(S)$ but we can do that here too $full_n(S) = hom(T_n(S), S)$. We'd have a map from the associative algebra to here. But now we should have a map of an ∞ operad. What's going to happen, you have a resolution $Rtriv$, for any tree you'll have a bunch of compositions. Then you cross with $full_n$, and you can organize the tensor product and you'll get a natural map

$$Rtriv(t) \otimes full_t(S) \to full_{N(t)}(S)$$

where $full_t := \otimes_v full_{|v|}$. We change the usual definition with this $Rtriv(t)$. Since $Rtriv$ is isomorphic to the trivial operad, more or less the category of operads in the usual setting is equivalent. You have the usual version $Assoc(n) = \mathbb{Z}$ and then you have this still satisfies the definition. We should call the version before an ∞-operad, and $Assoc$ is an ∞ operad. We want a map from $RAssoc$ to $full(S)$, and this will be that we endowed our S with an algebra structure. How to do this? Again by counting obstructions. We evaluate these spaces and prove they have no negative cohomology. Then all of our things will behave like usual Abelian groups, we'll be able to cut off below $t = 0$. You can solve the problem of the map to $t \leq 0$-$full(S)$.

Maybe I will skip this and give the answer, which is that $R^\bullet hom(T_n(S), S) = H^\bullet(G))^{\otimes n}$. How do you do this? It's a tricky question. You need a 2-algebra structure or maybe 3-algebra structure (somewhere) and then this will be this kind of operad, but we get around this with the cutoff.

So this finishes the story and the algebar is more or less unique.

[You take the *dg* category $\mathscr{A}$ on $G \times E \times E \times \mathbb{R}$, then we define a monoidal structure. You want to define a convolution for integration along G. There is an infinity version of that. What is the problem? Why isn't it strict?] Well, it's a problem with open sets not being products of open sets.

I need a dg structure to get richer things that maps in the homotopy category. There are the operadic school and the category school.

Let me demonstrate finally what we'll need in the future. Eventually we'll recover $O(n) \times \mathbb{Z} \hookrightarrow G$ where $\mathbb{Z}$ is the center of G (or a kernel of the cover, that's actually what I want)

Then you have an obvious map $O(n) \to Sp(2n)$, wher you act on $E \times E^*$ on the first factor. If you take the preimage of the image of $O(n)$ in $\widetilde{Sp}(2n)$ you get $O(n) \times \mathbb{Z}$. Let's actually use $SO(n)$

Then the question, we can compare $SO(n) \times E \times E$, we can restrict $S|_{SO(n)} \in D_{>0}(SO(n) \times E \times E \times \mathbb{R})$. (this restriction is actually to $SO(n) \times E \times E \times \mathbb{R}$). We can also look at the constant sheaf $\mathbb{Z}_{(g,e_1,e_2)|_{e_2 = ge_1}} \otimes \mathbb{Z}_{t \geq 0}$, call this Σ. Now $S|_{SO(n)} \cong \pi^{-1}\mathcal{L} \otimes \Sigma$ where π is the projection $SO(n) \times E \times E \times \mathbb{R} \to SO(n)$ and $\mathcal{L}$ is the unique nontrivial local system on $SO(n)$ with monodromy -1. Now, still, not today.