MICROLOCAL CATEGORIES
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I made a mistake at the end of the last lecture but I almost finished writing a file where it misses some proofs but in much more detail than I can provide on the blackboard. Today I will go to Chicago so maybe not. Over the weekend I will make this available. Maybe I won't correct it. It's just in the very end. Instead, so today let me write something for $O(d)$ and we did it only for $SO(d)$. We'll quantize all embeddings of the symplectic ball. Then if I have a Lagrangian submanifold, how to describe the category of all objects supported on this Lagrangian. I won't finish today. Maybe we can have a lecture after Thanksgiving. Let us do it this way. I'm not going to describe what I said, this category is hard to describe, but maybe I'll talk about the classical limit. Then the same as in the Fukaya category, there is a nontrivial obstruction theory. So first of all let me generalize not even to $O(d)$ but if you have this universal cover $\widetilde{Sp}(2D)$, you have a map $GL(\mathcal{D}) \to Sp(2D)$. Call the preimage of $GL(\mathcal{D})$ H, it is $GL(D) \times \mathbb{Z}$.

We have our sheaf $\mathcal{S}$ on $D_{>0}(G \times E \times E \times \mathbb{R})$ and we would like to describe $\mathcal{S}|_{H\times E\times E\times\mathbb{R}}$. It will be a local system on H times something very trivial. So we need to write, I don't know how to, you need these projections $p : H \times E \times E \times \mathbb{R} \to H$. For some local system $\mathcal{L}$ it should be, the restriction should be $p^{-1}\mathcal{L} \otimes \mathbb{Z}_{\{(h,e_1,e_2,t)|e_2=he_1,t\geq 0\}}$. The only novelty is about the local system. So $GL(D)$ as a topological space is isomorphic to $GL_{>0}(D) \times \{1, -1\}$ so you have a projection to $GL_{>0}(D)$ (depending on a negative element) which accepts a homotopy equivalence from $SO(D)$. Here we have only one nontrivial local system $\mathcal{L}$ and on $GL(D)$ you put $\mathcal{L} \sqcup \mathcal{L}$ and call it $\mathcal{L}_{GL(D)}$. Then the extension to $GL(D) \times \mathbb{Z}$ you do a shift for n by $2n$ units. So $GL(D) \times n \hookrightarrow GL(D) \times \mathbb{Z}$ and here we define $\mathcal{L}_n = \mathcal{L}_{GL(D)}[2n]$. This depends on a choice of generator of $\mathbb{Z}$, it could be negative.

This is going to depend on the action of the center. You take rotation by ninety degrees on one pair of variables in $SP(2D)$. In the cover the fourth power will be a generator, and it's the fourth power of the Fourier transform, which leads to the shift by 2. So you can prove, you take $\sqcup_n \mathcal{L}_n \in \mathcal{L}oc(H)$ and call this $\mathcal{L}_H$. So you have $H \times E \times E \times \mathbb{R}$ injects by i into $G \times E \times E \times R$ and then $i^{-1}S \cong p^{-1}\mathcal{L}_H \times \Sigma[dim\ G]$. Here Σ is the trivial sheaf and H is the lift of $GL(D)$ to the universal cover.

There is another issue that I will porbably suppress. We should have an algebra structure induced on $i^{-1}S$. But you need to use $i^!S$ if you want an algebra structure. This will be similar but with a different shift, $i^{-1}S[dim\ H - dim\ G]$. Then by counting dimension you can see that it has only one reasonable monoidal structure.

Now we will use this result. With this restriction we will only need it for quantizing objects supported on the Lagrangian but we would like all symplectomorphisms. That's too hard but we can quantize all symplectic embeddings $\mathcal{F} : B_R \to T^*\mathbb{R}^n$ where B_R is the standard symplectic ball of radius R. Say $\mathcal{F}$ is a symplectic embedding. Maybe to not repeat the same things several times, we'll quantize families. Suppose you have $F \times B_R \to T^*\mathbb{R}^n$, call this ϕ, and for $f \in F$ we have $\phi_f \times B_R$ is a symplectic embedding. F is a smooth manifold.

To quantize means more or less the following. Let $B_R \hookrightarrow T^*\mathbb{R}^n$ be an embedding I. What does it mean to quantize it? I assume that B_R is smoothly extendable to the closure (or the ball is closed). To quantize, you have $I(\circ B_R) \subset T^*\mathbb{R}^n$, and you have $D(I(\circ B_R))$, the quotient $D_{\geq 0}(\mathbb{R}^n \times \mathbb{R})/\text{complement}(I(\circ B_R))$, and we'd like to associate this with our category $D(\circ B_R)$. It would be nice if this changes the singular support according to I.

Maybe we should discuss this category. It's impossible to work with the quotients. It's important to have the theorem that it can be realized as a left or right orthogonal complement. This is true in more generality, and so here you can say that this can be realized in this way, but for this ball you have a special argument using our quantization of the metaplectic group.

I want to say that $D_{T^*\mathbb{R}^n \setminus \circ B_R}(\mathbb{R}^n \times \mathbb{R})$ which is a subcategory of the whole subcategory $D_{>0}(\mathbb{R}^n \times \mathbb{R})$ and I want to prove that this embedding has both a left and right adjoint. The adjoint functor will encode symplectic cohomology. Think of this as the standard unit ball. This is given as $p^2 + q^{<}1$. The flow is given by linear equations. The flow of H is $\mathbb{R} \to \widetilde{Sp}(2D)$. I think it's clear how to do it. Then let us consider the restriction $Flow^!_H \mathcal{S} \in D_{>0}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$. I'll call the coordinates (a, q_1, q_2, t). The singular support of the restriction? We can read it off from the singular support of $\mathbb{S}$. The microsupport $MS(\mathcal{S}_H) = \{(a, b, q_1, p_1, q_2, p_2) : (q_2, p_2) = e^{aX_H}(q_1 - p_1), b = H(q_1, p_1)\}$ where b, p_1, p_2 are dual coordinates in the cotangent bundle.

If we Fourier transform with respect to the a variable we have $\mathbb{F}_a \mathcal{S}_H \in D(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$. Now b is the coordinate on the base. If I restrict to $0 \in \mathbb{R}$ for the flow I should get something like the identity. So $\mathcal{S}_H|_{0 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}} = \mathbb{Z}_{(q_1 = q_2, t \geq 0)}$. Restriction corresponds to integration over al of $\mathbb{R}$ for the Fourier transform. Call $\mathbb{F}_a \mathbb{S}_H$ by Σ_H, then $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by π then $R\pi_! \Sigma_H = \mathbb{Z}_{\{q_1 = q_2, t \geq 0\}}$ up to a shift. Shift Σ_H so that this shift is 0. We have some sheaf such that the integral over b will give us just the identity. Now what you can do, if you have any sheaf, I should use a completely new letter $\mathcal{Z} \in D(\mathbb{R}^n \times \mathbb{R})$ then I can take the convolution $\Sigma \circ_{\mathbb{R}^n \times \mathbb{R}} \mathcal{Z}$. This is in $D(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$. Suppose that the microsupport of $\mathbb{Z}$ is in (q, p) such that $H(q, p)$ is in a closed set I, then the usual support of $(\Sigma_H \circ \mathbb{Z})$ is contained in $I \times \mathbb{R}^n \times \mathbb{R}$.

We know how to do our projection, you just need to restrict, now we will define our factor as follows. Let $\Sigma \circ \mathcal{Z}$. Then we need to take a piece with eigenvalue [unintelligible]. So we can multiply by the constant sheaf $\mathbb{Z}_{(b,q,t)|b \geq 1}$ and integrate along b, and so $R\pi_!(\Sigma \circ \mathcal{Z} \otimes (\mathbb{Z}_{(b,q,t)|b \geq 1})$ is $\Pi(\mathcal{Z})$, and Π is an endofunctor on $D(\mathbb{R}^n \times \mathbb{R})$. So you have a transformation $Id \to \Pi$, and you have $R\pi_! \Sigma \circ \mathcal{Z} \otimes \mathbb{Z}_{b,q,t,b<1} \to \mathcal{Z} \to \Pi \mathcal{Z}$. Then the first term is in the left orthogonal complement. You need [unintelligible]. Maybe you can just do it a simple way.

Now we should start thinking about quantizing families of symplectomorphisms, we'll do that next time.