MICROLOCAL CATEGORIES
DMITRY TAMARKIN

GABRIEL C. DRUMMOND-COLE

[Came in late]

Talking about $S \in D_{>0}(G \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R})$. $Sp(2D)$ acts on $T^*\mathbb{R}^D$.

We have $S|_{e\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R}} = \mathbb{Z}\Delta \boxtimes \mathbb{Z}_{t\geq 0}[D]$. We think of $SO(D)$ as sitting inside $Sp(2D)$ and this lifts to the universal cover $\widetilde{Sp}(2D)$ so we can talk about $S|_{SO(D)\times\mathbb{R}^D\times\mathbb{R}^D\times\mathbb{R}}$.

$\Sigma \in D_{>0}(SO(D) \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R})$, $\Sigma = \mathbb{Z}_K \boxtimes \mathbb{Z}_{t\geq 0}[D]$ for $K = \{g, x_1, x_2 | x_2 = gx_1\} \subset SO(D) \times \mathbb{R}^D \times \mathbb{R}^D$.

Claim: $S \cong \Sigma \otimes \mathcal{L}$ for some local system $\mathcal{L} \in Loc(SO(D))$ with $\mathcal{L}|e \cong \mathbb{Z}$.

You can define a functor that will give you this $\mathcal{L}$. Let me call the group $SO(D)$ by H and the space $\mathbb{R}^D$ by E, and then I need a convolution $D(H \times E \times E \times \mathbb{R}) \overset{\circ}{\times} D(H \times E \times E \times R) \xrightarrow{\circ} D(H \times H \times E \times E \times \mathbb{R})$, where this is convolution of the second E of the first factor and the first E of the second factor. You also have an embedding $D(H \times E \times E \times \mathbb{R}) \xrightarrow{i} D(H \times H \times E \times E \times R)$ where h embeds as h, h^{-1}.

Let me also define $\tilde{\Sigma} = \mathbb{Z}_{\tilde{K}} \boxtimes \mathbb{Z}_{t\geq 0}$ where $\tilde{K}$ is the opposite: $\{g, x_1, x_2 | x_1 = g(x_2)$.

Now we can look at $i^{-1}(S \circ \tilde{\Sigma})$. From the singular support, you can see that it is a locally constant sheaf on $H \times \Delta_E \times \mathbb{R}_{\geq 0}$. Restricting on this diagonal, where it is supported, it is locally constant. Since $\Delta_E \times \mathbb{R}_{\geq 0}$ is contractible, you will get a local system on H. If you do the same thing with Σ you will get the inverse system. Therefore, since we know this one, we can get our original sheaf S by applying the inverse functor. Call this one $G(S)$ and $F(S) = i^{-1}(S \circ \Sigma)$ so then $F \circ G \cong \mathrm{id}$, and $S \cong F(\mathcal{L} \boxtimes \mathbb{Z}_\Delta \boxtimes \mathbb{Z}_{t\geq 0})$. Which local system is this? We can prove that the fiber at the unit is $\mathbb{Z}$ since $\mathcal{L} \otimes \Sigma|_e \cong S_e$. There are two of these, the trivial one and the twisted one. If we restrict to $SO(2)$ we can see this. So look at $\mathcal{L}|_{SO(2)}$ where $SO(2) \subset SO(D) \subset \widetilde{Sp}(2D)$. You could just as easily inject in $SU(2)$ in $SU(D)$ in $\widetilde{Sp}(2D)$. So we might as well assume that $D = 2$. Now $SU(2) \subset \widetilde{Sp}(4)$ is a homotopy equivalence, by accident, so now we need to look at $SU(2) \times E \times E \times \mathbb{R}$. Now we really need to construct the sheaf here and see what happens when we restrict to $SO(2)$. We're stuck, it's really a concrete computation.

We have $\mathbb{R}^4$ with coordinates (q_1, q_2, p_1, p_2). It's useful to introduce the coordinates $Q = q_1 + iq_2$ and $P = p_1 - ip_2$, and then $\alpha = \mathrm{Re}(PdQ)$. If you have a matrix $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$, then

$$\begin{pmatrix} \tilde{Q} \\ \tilde{P} \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}.$$

Now we will quantize using generating functions. We need to do it on two charts separately. One, U_1, will be $a \neq 0$ and the other U_2 will be $b \neq 0$. They will mismatch by a local system. You need to choose a pair of independent coordinates. If $a \neq 0$ you can choose $\tilde{Q}$ and P (you will have $\tilde{Q} = aQ + bP$ and $P = \frac{\tilde{P} + \bar{b}Q}{\bar{a}}$). You need to compute $\mathrm{Re}(\tilde{P}d\tilde{Q} + QdP)$ which is necessarily $dS(\tilde{Q}, P)$. This can be computed, we have $S(\tilde{Q}, P) = \mathrm{Re}\frac{2P\tilde{Q} - bP^2 - \bar{b}\tilde{Q}^2}{2a}$

If $b \neq 0$ you can use Q and $\tilde{Q}$ [!]. You have another formula and you define $d\Sigma = \mathrm{Re}(\tilde{P}d\tilde{Q} + PdQ)$ and $\Sigma = \mathrm{Re}\frac{\bar{a}\tilde{Q}^2 + aQ^2 - 2Q\tilde{Q}}{2b}$.

Now let us see what kind of sheaves we can obtain from these functions and what we get on the overlap.

The idea is that first I can consider $t + S(\tilde{Q}, P) \geq 0$ with $\mathbb{R}(\tilde{q}_1, \tilde{q}_2, p_1, p_2) \times \mathbb{R}$. We need to do a Fourier transform. If you take the convolution with $t - PQ \geq 0$, overall it will look as $t + S(\tilde{Q}, P) - PQ \geq 0$. Take the constant sheaf on this set in $\mathbb{R}^6_{Q\tilde{Q}P} \times \mathbb{R}$, and then you push forward along P to $\mathbb{R}^4_{Q\tilde{Q}} \times \mathbb{R}$. I forgot the group variable, you should also multiply all factors by $U_1 \subset SU(2)$ where $a \neq 0$.

Now we should compare it to the other quantization $\mathbb{Z}_{t+\Sigma(Q,\tilde{Q})\geq 0}$. Call these $\mathcal{S}_1$ and $\mathcal{S}_2$.

[Quantization of a group is a sheaf on $G \times E \times E \times \mathbb{R}$, and if you restrict to an element you should get a transformation of E. An object on $E \times E \times \mathbb{R}$ gives you an endofunctor on $E \times \mathbb{R}$. It's a way of encoding an action of G on $E \times \mathbb{R}$.]

Now we need to compare two objects. Both can be restricted to the overlap $U_1 \cap U_2$. We will see they differ by a local system. You remove two big circles from S^3, which are linked but do not intersect. You need to complete the square, if a and b are nonzero, and you get the following formula.

$$S(\tilde{Q}, P) - PQ = \mathrm{Re}\left\{-\frac{b}{2a}\Pi^2 + \Sigma(Q, \tilde{Q})\right\}$$

where $\Pi = (P + \frac{a}{b} - \frac{1}{b}\tilde{Q})^2$.

This overlap is S^1 times a disk. If $a \neq 0$, maybe you can just, what you need to do, if $b \neq 0$, you can instead of projecting along P, you can project along Π. So restricting to the intersection the first sheaf is $R\Pi_!\mathbb{Z}_{\left\{\mathrm{Re}\left(t - \frac{b}{2a}\Pi^2 + \Sigma(Q,\tilde{Q})\right) \geq 0\right\}}$. We can focus to where we project, first restrict yourself to $R\Pi^0_!(\mathbb{Z}_{t - \mathrm{Re}\frac{b}{2a}\Pi^2 \geq 0})$ and we can look at $U_1 \cap U_2 \times \mathbb{R}^2_\Pi \times \mathbb{R}_t \xrightarrow{\Pi^0} \mathbb{R}$.

I don't have time, this is a good exercise, you have $U_2 \xrightarrow{Arg\frac{a}{b}} S^1$ and the pushforward is $\alpha^*\mathcal{L}_{S^1} \boxtimes \mathbb{Z}_{t\geq 0}$. So $\mathcal{S}_1 \cong \mathcal{S}_2 \otimes \alpha^*\mathcal{L}_{S^1}$.

How can you glue this with one sheaf? You need to choos it to be $\mathcal{S}_1$ on one chart and $\mathcal{S}_2$ on the other chart or vice versa. Then you need to restrict it to $SO(2)$, and you realize this as $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$. You have two choices because we don't know which one to shift. The neighborhood of 1, when $b \neq 0$, we should have the unchanged sheaf. So we should change $\mathcal{S}_1$

to $\mathcal{S}_1 \times \mathcal{L}$, and we need to restrict to $SO(2)$, and if you restrict you will get what I said. This can be computed, and you will get the formula $\mathcal{L}_{S^1} \otimes \mathrm{Re}\mathbb{Z}_{t+\mathrm{Re}(P(\tilde{Q}e^{-i\phi}-Q))>0}$.

I completely forgot the homological shift. They will work out together. This one, you can prove that this is t plus the Fourier transformation, and then you project on P you'll get the delta function on the difference $\tilde{Q}e^{-i\phi} - Q$.