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Good afternoon and welcome to this course. The title is Frobenius Manifolds, Quantum Cohomology, and things like that.

This subject was born 20 years ago in theoretical physics. I feel that it's incredible, the creation of such a big theory, all of this happened in the last twenty years. The only thing I will be able to do for you is to give an intorduction to this story. An introduction can essentially be given in two different modes. Assuming you are sufficiently versed in classical algebraic geometry, I could have introduced you to a couple of key constructions and theorems, and by this time the term would be over. It's not very practical. I'll try to present this subject from two complementary viewpoints, one, the broad view of what is important, or two, problems that are solved with the general theory or still unsolved. I can't then give you proofs, but there is much literature, and I'm happy to supply you with references.

So I will start now with a historical background. I have focused this on a general question that can be stated as follows: how many solutions has a(?) given (system of) equations? Actually, right away, it's better to keep in mind that considering an isolated equation is unreasonable from a mathematical viewpoint, and it's always better to consider families, which will depend on an integral or real parameter.

Okay, so let's start with the simplest thing. The equation is one polynomial in degree n in one variable: $f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0$. Look at how many ideas this evolved. The "answer" to this story is that there are n solutions. There are a lot of provisos and subtleties. One is where our a_i and our solutions are presumed to be. The answer was first the invention of complex numbers, and then later with the ideas of algebraically closed fields. The theory like that is true for a_i and x in an algebraically closed field.

Second, how do you count solutions? This led to the notion of counting with multiplicities. How do you count multiplicity? First, prove that if you have a root $x = \rho$, then $f(x)$ is divisible by $(x - \rho)$. If you take the maximal power of $(x - \rho)$ by which it is divisible, a_ρ, then this will be the multiplicity. Then you count the number of roots by adding the multiplicities. You must produce an environment where this has a nice stable independence. You need an environment (algebraically closed field) and then a way to count correctly.

A variation, deform things. Rather than counting multiplicity, ask about a "typical" equation, which means approximately generic values of coefficients. Already on this historic example, you have explicit formulas for quadratic then cubic roots, then the statement starting with degree more than four, there are no such formulas. There is a whole lot of mathematics around this one simple equation.

We've passed into the 19th century. This took a lot of time.

Now take two variables in two equations $f_m(x,y) = 0$ and $g_n(x,y) = 0$. Then you had Bezout's theorem, saynig that the number of solutions is mn. Already you should keep in mind the lessons of the simple story. What environment? How do you count solutions? Here solutions may go away to infinity. For example $x + y + 1 = 0$ and $x + y + 2 = 0$. In principle there is one solution, if you deform the linear equation. If you don't deform then you should be aware that $f = 0$ and $g = 0$ can be tangent at a point. In complete generality in any dimension, Serre, in my memory, said you should take the alternating sum of certain dimensions of Tor sheaves. Nothing like this was known then.

There is a nice way to obtain this formula intuitively. Let me imagine $f_m = 0$ and $g_n = 0$. Then I will independently deform the coefficients of f and g, but in a very untypical way. So I'll deform $f = 0$ to $\prod_{i=1}^m f_i = 0$ and g to the product of n independent linear pieces, and then the mn points are the mn intersections of these lines. In order to catch the runaway solution is to compactify and include an ∞ to get to the projective plane. One of the simplest ways to compactify it is to put it on a quadric in $\mathbb{P}^3$. Such a quadric is a product $\mathbb{P}^1 \times \mathbb{P}^1$, so then I'll get something like

$$f(\underset{1}{\overbrace{x_0, x_1}^{m}}; \overbrace{y_0, y_1}^{m_2}) = 0$$
$$g(\underset{1}{\overbrace{x_0, x_1}^{n}}; \overbrace{y_0, y_1}^{n_2}) = 0$$

and then the "Bezout" theorem will give us $m_1 n_2 + n_1 m_2$; each projective will give its own intersection. This "intersection index" now depends on multiple degrees. You can imagine the same picture as before. Such an equation can be deformed to m_1 lines that are all parallel to each other, and so on, so that they only intersect as described.

Instead of a quadric, we can look at a cubic in $\mathbb{P}^3$ A cubic is an interesting animal. There are twenty-seven lines on a cubic. Their configuration is always the same. It's incredibly beautiful with the E_6-group as a group of symmetries. This is the first question that has some similarity to the questions that were asked in quantum cohomology. This is all part of what was called "enumerative algebraic geometry."

What will be important for me is to explain to you that algebraic geometry did not generate this, and we needed physicists to push us to this.

A very interesting question in the history of interactions between geometry and physics was in 1991. There was a conference organized by a combined group of mathematicians and physicists, who were interested in quantum string theory. The question that was discussed there was similar to this one. Consider a quintic sitting in $\mathbb{P}^4$.

If you start counting lines, there are a finite number of lines, the number was known. The number of general conics was also counted. Then two groups conflicted about how many curves of degree three lie on such a quintic. Candelas was a physicist and pronounced that the number was 317206375, using extremely sophisticated physics, not based on mathematics. The group of mathematicians, I'll write Ellingsgood and Strøm, produced another answer, 2682549425. This was embarrassing, who was right? And the eventual answer, a few weeks later, was that physics wins. You should put this kind of problem in a larger context. We started with lines, then conics, then cubics, so now, how many rational curves of any degree are there (typically)? Then, what does it mean, typically, and when does typicality agree with reality? So then if N_d is the number of rational curves of degree d on a quintic in $\mathbb{P}^4$, this turned out to be an extremely fruitful question, but to show what kind of answer they got, they got the answer for any d. They got a power series, whose coefficients were essentially these N_d.

So now we must return to the nineteenth century and an earlier example: you of course know or have heard that any integer can be represented as a sum of four squares. Jacobi got a precise formula for it. The number of $(x, y, z, u) \in \mathbb{Z}^4$ so that $n = x^2 + y^2 + z^2 + u^2$. it is a finite, nonnegative number, and there is this remarkable formula for it. Take the divisors d of n which are not divisible by 4. This number is $8 \sum_{d|n, 4\nmid d} d$

This is an example, $f(n)$ and $g(n)$ are everywhere defined and computable. You can write a program for computing f and g. You can ask why these would give the same answer. One environment in which things like this can be proved, you have many equivalent programs where you change things by elementary steps. Maybe there are elementary steps, but this is not true for all f and g. It's certainly wrong that two programs that compute the same function are always obtainable from one another. The fantastic discovery by Jacobi was that you should consider first a generating function. Denote our number by $q(n)$, and the one on the right by $\sigma_{(1)}(n)$. You should produce a generating function for this number is to write $\sum_{-\infty}^{\infty} e^{2\pi i n^2 z}$. For four squares, take the fourth power of this and you will get $\sum q(m)e^{2\pi i m z}$. So then you can produce a generating series $\sum \sigma_{(1)}(m)e^{2\pi i m z}$. Now you are considering functions in the upper half plane. You see that your function is periodic with period one, good, and that there is a kind of summation formula that produces a relationship between such sums and $\sum \sigma_{(1)}(m)e^{-2\pi i m \frac{1}{z}}$. You prove this, and then you show something about its analytic behavior when z goes to a rational number. The function is now determined by its analytic properties. Then you look at another such function, but using different tools, you can prove the same properties for it. Then you should prove a uniqueness theorem. You don't calculate a single coefficient. You proceed in a totally different way, but you achieve the result.

Why is this related to quantum cohomology? Candelas et. al. wrote a function $\sum N_d(\cdots)$, I will not write it exactly. Mathematicians had not looked at such problems in such a way so that putting things all in order you got an analytic function, no one had done it for enumerative algebraic geometry. This story was not only the beginning of quantum cohomology, but also the mirror phenomenon.

I will use another example due to Kontsevich, which leads in a more straightforward way than the context of Candelas and company. Maxim Kontsevich has asked the following question: (I was probably wrong when I said that they weren't considering generating functions. The most famous example, you consider a variety over F_{q^n}, and count the intersections, and then this is the same as the intersection index of the diagonal in $X \times X$ and $\Gamma_{x \mapsto x^{q^n}}$, and Lefschetz thought that these should assemble to a zeta function.)—the quantum cohomology story was of different type. Kontsevich answered this question, and this is already a door to quantum cohomology. Let's consider the question: our ambient variety is $\mathbb{P}^2$, over any algebraically closed field, think $\mathbb{C}$, if you ask for lines between two points, there is only one. Then we can consider conics. The same story. This time to make things finite, we must ask them to pass through five points. The answer is again one, through a generic system of five points. What is the correct next result? We can ask cubics to pass through ten points or something like that. The thing is to remain in the domain of rational curves, and if a cubic is rational, it has one double point? How many points should it pass through so that there is a finite number, and what is the answer? Let's call it N_3, and we also have N_d for a rational curve of degree d, and generically it's $3d-1$ points that must be passed through, so now what is N_d, the number of curves through $3d-1$ points?

I can't find it in my notes. But the number can be expressed as a sum, something like:

$$N_d \overset{(?)}{=} \sum_{k+\ell=d, k,\ell\geq 1} N_k N_\ell k^2 \ell[?\binom{3d-2}{3k-1} - ?\binom{?}{?}]$$

I'll give you the exact formula next time if I remember. There's an intuitive explanation via deformations. We imagine that in the totally generic case, there is this curve with enough double points passing through $3d-1$ points. Instead imagine that we are counting a similar story, but our curves consist of two components of the same nature. If we distribute these points in various ways between these two components, we'll get something that will produce the product $N_k N_\ell$. But then there is some combinatorics, and that gives the other number. That can be done simply if you think of how it can be done, but what is much less obvious is that this peculiar recursive relation has an interesting and totally nontrivial and at that time quite new interpretation in terms of a generating function. What you do is something like that, you produce the generating function $\sum \cdots x^a y^b z^c$, which is a formal series. It converges somewhere, at that time I think that was not yet known. I'll say right away that x, y, and z are linear coordinates on the cohomology space of $\mathbb{P}^2$, which is generated by the fundamental classes of a line, point, and $\mathbb{P}^2$. So this is a formal function on the cohomology of your ambient space. This cohomology has an intersection index, a bilinear form, so that $\Delta_1^2 = 1$ and everything else is more or less trivial. Therefore you can lift indices, so if I write $\sum x^i \Delta_i$, then you have this formal function, and what you do is introduce a new multiplication $\Delta_i * \Delta_j = \sum \phi_{ijk} g^{k\ell} \Delta_\ell$. This obviously defines a structure of commutative algebra on the cohomology of $\mathbb{P}^2$ with coefficients in (tensored with)$\mathbb{C}[[x_0, x_1, x_2]]$. Remarkably, the associativity is due to the formula of Kontsevich. Commutativity is clear because the derivatives commute. The recursive formula that appeals to your geometric intuition, and says to look at a degenerate condition. If it was not for physicists, Dijkgraf, Verlinde, Verlinde, and Vafa [sic?] I don't know, it would have been hard to discover it. Now it encompasses most of algebraic geometry. This says a motive of any algebraic manifold is an algebra over an operad that comes from other algebraic manifolds (or stacks). I hope to be able to explain this much later. This is the first example of things like that. This all happened in the last twenty years.

I'll try to present to you this story going from the general overview to specific examples, and I hope that you will enjoy it. Are there any questions? No questions? Everything is crystal clear? Thanks.