## YURI MANIN

GABRIEL C. DRUMMOND-COLE

I have been considering the role of the dimension axiom, and have shown that it's a homogeneity condition that can be described differently in terms of the quantum multiplication, which translates into the $F$-manifold and Frobenius manifold situation. So the dimension axiom in quantum cohomology traslates into a homogeneity with weights of the quantum potential $\phi^{V}\left(x_{a}\right)$ which translates using Euler's idea into, well, the most essential part will be an eigenvector for the Euler vector field $E=\sum_{a}\left(i-\frac{\left|\Delta_{a}\right|}{2}\right) x_{a} \partial_{a}+\sum r^{b} \partial_{b}$ for $\left|\Delta_{b}\right|=2$, where $\sum r^{b} \Delta_{b}=-K_{V}$. Based upon this, we transform it into the notion of Euler fields on $F$-manifolds, so we have
(1) the compatibility with $\circ, P_{E}(X, Y) \equiv \underbrace{d_{0}}_{\text {weight }} X \circ Y$ which is the same thing as $\operatorname{Lie} e_{E}(\circ)=$ $d_{0} \circ$.
(2) If a flat structure is given, then $\left[\mathscr{T}_{M}^{f}, E\right] \subset T_{M}^{f}$ so I get the notion of the spectrum of $-\mathrm{ad} E$ on flat vector fields, and
(3) if a metric $g$ is given, then $\operatorname{Lie}_{E}(g)=D g$,
and so we have the full spectrum of $E$ as $\left(d_{0}, D\right.$, spectrum of $\left.-\operatorname{ad} E\right)$ (this spectrum depends on the flat structure).

Let's look at an example. On a semisimple (simply connected) $M$, we have ( $u_{i}, \delta_{i}$ ) our local canonical coordinates, an easy exercise says that any $E$ is of the form $\left(d_{0} u_{i}+c_{i}\right) \delta_{i}$ so the spectrum of $E \circ$ on $\mathscr{T}_{M}^{f}$ is $\left\{d_{0} u_{i}+c_{i}\right\}$. Normally we renormalize to make $d_{0}=1$. It doesn't matter if I'm calculating at one point. Suppose I have a specific $E$ that comes from another description, but simultaneously I want to check that it is somewhere semisimple. I can write $E \circ\left(\begin{array}{c}\Delta_{0} \\ \vdots \\ \Delta_{N}\end{array}\right)=\mathcal{E}\left(\begin{array}{c}\Delta_{0} \\ \vdots \\ \Delta_{N}\end{array}\right)$, and if I put the $x_{a}=0$ except for divisors. All that remains is quantum multiplication by the canonical class. This is usually a very easily calculable matrix $\mathcal{E}$, suppose that I am so lucky that $\mathcal{E}$ clearly has a simple spectrum. Then at this point, at a generic point, I will have semisimplicity. So all I will need is the quantum multiplication modulo the hard parts. It turns out that quantum cohomology of projective space is semisimple.
... in Saito the canonical coordinates are easy and in quantum cohomology the flat coordinates are easy.

Let me continue the story about Saito structures. We have $\left(z_{a}, t_{b}\right)$ coordinates in $N$ and then we project just to $t_{b}$ coordinates on $M$ via $p$, and we maybe fix a point $0 \in M$, and then we have a function $F=F(z, t): N \rightarrow \mathbb{C}$, and then we have the critical space $\mathscr{C}=\left\{d_{z} F=0\right\}=$ $\left\{\frac{\partial F}{\partial z_{a}}=0\right\}$ which has the projection $p_{\mathscr{C}}$ to $M$. An essential part of the construction is the map $\mathscr{T}_{M} \xrightarrow{s} p_{\mathscr{C} *}\left(\mathscr{O}_{\mathscr{C}}\right)$ which takes $X \mapsto \bar{X} \in \mathscr{T}_{N}$ locally, and then apply $\bar{X} F$ and then reduce
this modulo $\mathscr{C}$ to get $\left.\bar{X} F\right|_{\mathscr{C}}$. This final thing depends only on $X$ and we'll denote it $s(X)$. A very essential assumption is that the initial data is such that $s$ is an isomorphism of $\mathscr{O}_{M^{-}}$ modules. This produces a multiplication on $\mathscr{T}_{M}$. Either take $X \circ Y=s^{-1}\left(\bar{X} F \bar{Y} F \bmod \mathcal{J}_{F}\right)$ or define $\left.\overline{X \circ Y} F\right|_{\mathscr{C}}=\bar{X} F \bar{Y} F \bmod \mathcal{J}_{F}$. You get the multiplication first. There is a very specific construction of compatible flat structures together with metrics for which, it is in a sense, it is an existence theorem but it comes naturally in most environments. I will not prove it but just give you formulas that you can use. These are useful not for theorems but for mirror symmetry calculations.

You could take your $F$ to be some kind of unfolding of an isolated singularity but that is not as interesting, but rather a Laurent polynomial. So $\mathbb{P}^{n}$ mirrors correspond to unfolding $F\left(z_{0}, \ldots, z_{n}, 0\right)=z_{0}+\cdots z_{n}+\frac{1}{z_{0} \cdots z_{n}}$, and there is also some sort of explanation of why this should be the thing to unfold but they are not satisfactory, so Sabbah and Douai have done this in the most interesting cases. The isolated case and the Laurent case and very different. What kind of functions should be considered on the Laurent side, that is not clear and is a very interesting question. This was results of clever guesses and analogies, although at first these led to singularities. So sometimes $H^{*} V$ can be represented as the Jacobi ring of a polynomial, if you take an appropriate generating set for homology, $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]\left(\frac{\partial F}{\partial x_{i}}\right)$. These will have rational values for the Euler field, and thus we should think that there are some stacks on the other side. For projective space $\mathscr{F}$ is very easy, $H^{*}\left(\mathbb{P}^{r}\right) \cong \mathbb{C}[z] / z^{r+1}$ which is isomorphic to $\mathbb{C}[z] / \frac{\partial z^{r+2}}{\partial z}$, and we could unfold $z^{r+2}$ and calculate the Saito picture and we get something like a projective space whose dimension of intermediate subspaces lie between 0 and 1 instead of 0 and $r+2$. This happens in analysis in $C^{*}$ algebras, and I always wanted to follow up on this analogy. There is some freedom in choosing $E$ or flat structure, but I could not establish an isomorphism.

Now Saito's construction of compatible flat metrics, the general scheme is applicable in the general case, is this:

Denote by the Hessian $\operatorname{Hess}(F)$ the restriction to $\mathscr{C}$ of the determinant of ( $\frac{\partial^{2} F}{\partial z_{a} \partial z_{b}}$ ) multiplied by $\left.d z_{1} \wedge \cdots \wedge d z_{b}\right)^{2}$ which is a section of $L^{2}$ where $L$ is the vertical volume forms $L=\left.\Omega_{N / M}^{v o l}\right|_{\mathscr{C}}$. If I change my coordinates, then the we multiply and divide by the same Jacobean, so whatever you calculate in local coordinates doesn't depend on those coordinates. Locally in $\mathscr{C}$ I have the hyperplane of the first coordinates and then the second derivatives. Then I will define a 1 -form $\epsilon$ on $M$. I can define $i_{X}(\epsilon)$ where $X$ is a vector field on $M$. Now $X$ I can replace by some local function on $\mathbb{C}$, let $X=s^{-1} f$ where $f \in \mathscr{O}_{\mathscr{C}}$. Then I'll take, well, over $U$ sits, in a generic case you have an unramified covering, you have $\rho_{i}$ over a point.

$$
\sum_{i=1}^{\operatorname{dim}} \frac{f\left(\rho_{i}\right)}{\operatorname{det}\left(\frac{\partial^{2} F}{\partial z_{i} \partial z_{j}}\right)\left(\rho_{j}\right)}
$$

which is not completely invariant but invariant under the unimodulars, [unintelligible].
Then I define $g(X, Y)$ by $i_{X \circ Y} \epsilon$. This will be some kind of quadratic form, invariant with respect to multiplication. But generally you cannot expect nondegeneracy or flatness. Then Saito's definition, a primitive form $\omega \in \Omega_{N / M}^{\max }$ is an $\omega$ so that if $\left(z_{1}, \ldots, t_{b}\right)$ are unimodular, then the metric $g$ is flat. If you manage to find such a form, and in examples it is simple, for example, in the Laurent example we could use $\wedge \frac{d z_{i}}{z_{i}}$. A formula like that is often easier for explanations but better for calculations.

The intrinsic but not universal way to do this (and this doesn't always work in any natural context I know), you have a multiplication on $p_{\mathscr{C} *}\left(\mathscr{O}_{\mathscr{C}}\right)$ and transfer the product to $\circ$ on $\mathscr{T}_{M}$ via the isomorphism. So similarly, you connect $\mathscr{T}_{M}^{f}$ to something that has a notural flat structure and metric, something produced out of topology, out of the fibers of the unfolding of the function. You are considering the usual cohomology of the fibers but fibers are not generally compact, so you want, usually, to represent these by vanishing cycles of middle dimension. The usual picture that one draws is, you have a quadric as a generic fiber of $F$ and then it degenerates and becomes a cone, which means some cycle goes to 0 so that everything degenerates to a cone. This is a vanishing cycle. You map $\mathscr{T}_{M}^{f}$ isomorphically to homology generated by vanishing cycles, and it has some usual features of homology or cohomology of a real manifold. There is a pairing, there are constant coefficients, there is the intersection form, and there are formulas from Grothiendieck. Everything is obtained out of a geometric picture. This is not a universal recipe. When this works you get compatible structures by putting these things together.

Let me show you the results of concrete calculations for $\mathbb{C}[z] /\left(z^{n}\right)$.
Example 1. Unfolding of $F(z, 0)=z^{n+1}=\mathbb{C}[z] /\left(\frac{\partial z^{n+1}}{\partial z}\right)$. A basis consists of $1, z, \ldots, z^{n-1}$ $\bmod z^{n}$, and the usual construction of unfolding is to take $F(z, a)=z^{n+1}+a_{1} z^{n-1}+\cdots+$ $a_{n-1} z+a_{n}$. We have $(z, a)$ projected te $a$, and then the lift of $\frac{\partial}{\partial a_{i}}=\frac{\partial}{\partial a_{i}}$, and applying this lift to FI get $z^{n-i}$, so $\frac{\partial}{\partial a_{i}} \frac{\partial}{\partial a_{j}}=z^{2 n-i-j} \bmod \left(\frac{\partial F}{\partial z}\right)$. In particular $\frac{\partial}{\partial a_{n}}$ is the identity with respect ot $\circ$. The structure of $\mathscr{C}$ is not very obvious. The structure of $\mathscr{C}$ is the derivatives $\frac{\partial F}{\partial z}=(n+1) z^{n}+a_{1}(n-1) z^{n-2}+\cdots+a_{n-1}$, and you should consider where this is zero, the decomposition into linear terms $(n+1) \prod\left(z-\rho_{i}\right)$ where $\rho_{i}$ are algebraic functions of $a_{i}$. These are going to be symmetric so don't worry about the numbering of the $\rho_{i}$. Generically these will be pairwise distinct, and somewhere there will be double or triple roots. On the $A$-space there is a complicated picture of caustics, degenerations. The ring where you have the unramified story, this is a semisimple algebra, but when you degenerate you get more and more nilpotents.

If you wish to glue something more global you will have to understand that you are gluing this structure. Now wherever semisimplicity goes, outside of this degeneration, where $\frac{\partial F}{\partial z}$ has no multiple roots, the canonical coordinates are $u_{i}=F\left(\rho_{i}\right)$. This is the universal formula for any Saito structure (this is an exercise) where $\mathscr{C}$ is unramified over $M$. Therefore, we have one canonical vector field over the unramified story, $E=F\left(\rho_{i}\right) e_{i}$ where $e_{i} \in \mathscr{T}_{M}$, on $\mathscr{C}$ locally you have functions that are the identity on the slice $i$ and zero elsewhere. They should become idempotents with respect to my multiplication. The remaining freedom in the semisimple case is pretty minimal, you can multiply by something and add a constant.

Now, in this particular case the form $d z$ is primitive therefore the explicit formula that I gave produces an explicit Riemannian metric $g=\sum \frac{\left(d u^{i}\right)^{2}}{F^{\prime \prime}(z)\left(\rho_{i}\right)}$ which turns out to be flat and compatible with $\circ$. This is checked by pretty easy calculations. Then it turns out that recalculating the same $E$ in terms of $\left(a_{i}\right)$ it is $\frac{1}{n+1} \sum(i+1) a_{i} \frac{\partial}{\partial a_{i}}$, and you get the spectrum in this way.

Later one can find flat coordinates $x^{(i)}$, where you take the roots $w$ of $F(z, a)$ of degree $n+1$ near $\infty$. These will be $z$ plus a Laurent polynomial in pure negative degrees. Pass to an inverse function and you get, I'm not one hundred percent sure,...
[Can you explain the metric?] It should be on the bottom the determinant, and the product on top is just squares of canonical coordinates. I have given $i_{X \circ Y} \epsilon$. If you write it in this story you get what I have written.

I think I made a mistake when I copied my formula. It sholud be somethnig like

$$
z+\frac{x^{(1)}}{z}+\frac{x^{(2)}}{z^{2}}+\cdots+\frac{x^{(n)}}{z^{n}}
$$

I think I made a mistake and I've reversed $z$ and $w$.
There will be two days work, three plus three talks, and written papers on the subject of these talks. On Tuesday, November 16, Le, Gunningham and Chu, and here are subject matters, about $F$ and Frobenius manifolds, how they look locally, what happens when you lose semisimplicity. Probably you will not be able to explain proofs here but will write them in your paper. These three talks could be my lectures. The pages of these books should be copied and returned. The second three talks will be explaining these things. These three talks will be November 18, and this will be dedicated to reconstruction theorems. I will explain about generalizations before that to a much vaster framework, high genus, motivic framework, and then we will see that what algebro-geometric theory gives us is a picture where the numbers of the potential give us only a small part. Can we reconstruct? The talks on Thursday will be related to reconstructions. By November 18 I also want all six papers.

