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At the end of class on Thursday I will explain the grading event.

So let me remind you that we are looking for a smooth manifold V for maps that are supposed to have the properties that $I^V_{0,n,\beta} : H^(V, \mathbb{C})^{\otimes n} \to \mathbb{C}$, and β should be in $H_2(V, \mathbb{Z})/torsion$, and morally, $I^V_{0,n,\beta}$ (it's better for me to put this in angle brackets) $\langle I^V_{0,n,\beta}\rangle(\Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n})$ where Δ_a is a basis of representatives of $H^*(V, \mathbb{Z})/torsion$, is morally equal to the number of rational curves endowed with n marked points, pairwise different, stable, all that I will explain later, sitting in V and such that the point x_i sits in Δ_{a_i}. Basing myself on this idea, I've explained the list of properties of these numbers, and I have given all explanation except for the case that $\beta = 0$. I will explain this later. In particular, right now, I will focus my attention on the dimension axiom. This simply said that if such an expression $\langle I^V_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n)$ is nonzero, then restrictions on various arguments should be satisfied:

$$\sum(|\gamma_i| - 2) + 2[(K_V, \beta) + 3] = 2 \dim_{\mathbb{C}} V$$

This means I know the condition on the dimension of the incidence conditions for which $\langle I\rangle$ can be nonempty and noninfinite. Let me make three comments:

(1) How does this generalize?
(2) To what structure does the dimension axiom lead in a general context of Frobenius and F-manifolds?
(3) Answer to Le's question.

Le told me very reasonably that the axiom I formulated then, in the case that one of the Δ was 1 (in the cohomology ring), I said this should be the same as just forgetting this one, saying that, okay, this incidence condition is saying that my curve intersects with the whole V, but then I have a choice of x_n in arbitrarily many ways, but it follows with the divisor axiom that this is zero but it makes sense to include it. If one of the axioms is a divisor δ then $\langle I_{0,n\beta}\rangle(\gamma_1 \otimes \cdots \otimes \delta) = (\beta, \delta)\langle I_{0,n-1,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_{n-1})$.

If $\beta \neq 0$ then whenever 1 is an argument, $\langle I\rangle = 0$. Choos a δ with a nonzero intersection index, for an appropriate δ we'll get two different dimensions. [Something about when the identity axiom holds]

To address comment [1], $I^V_{0,n,\beta}(\Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n})$ will send $H^*(V)^{\otimes n}$ not to numbers, but to the class of cycles $(C, x_1, \ldots, x_n)$ with x_i and C in the right representatives and class, in the moduli space of all such curves, where this is understood appropriately.

Geometrically, I do not want to constrain myself to finitely many solutions. I want to consider finite dimensional families of solutions. This will become more transparent when I dualize. I have two cohomologies. I have maps of cohomology groups, [missed some]

Now about comment [two]. What is the generalization of the dimension axiom in the case that I am thinking about Frobenius manifolds and F manifolds? The numbers are Taylor coefficients, it turns out that the dimension axiom has a very good translation, and leads to an additional structure called the Euler field. This is something in addition to multiplication, metric, and so on.

So $F(x_1, \ldots, x_n)$ is a form of degree d if and only if $(\sum x_i \frac{\partial}{\partial x_i})F = \underbrace{d}_{\text{eigenvalue}} F$. I can generalize this, I can choose a number for each x_i, declare its weight, and calculate with respect to this, no problem, I can add weights to the factors of the sum. The reason is just the derivation formula, $Xf = d_1 f$ and $Xg = d_2 g$ then $X(fg) = (d_1 + d_2)fg$. Then on any commutative ring and any vector field I get a grading, maybe, but there may not be any such functions. So X defines a structure of a graded ring on the subring A_0 spanned by eigenvalue of X.

Consider the case $\mathbb{C}[x]$ and $X = \frac{d}{dx}$. Except for constants you have nothing. But if you consider $\mathbb{C}[e^x]$, then e^x is an eigenvalue, and there are all other eigenvalues. This is the reason we use, for divisors, an exponential version. One can also consider formal series. Then the effect will be the same.

Now if we look at the quantum cohomology potential $\Phi^V(x)$, we can split it into a cubic part which produces the classical multiplication and the quantum corrections $\Phi^V_{qc}(x)$, a series of monomials in x, then the dimension axiom means that the quantum corrections are homogeneous under assignment of certain weights,

$$E = \sum_a (1 - \frac{|\Delta_a|}{2}) x_a \partial_a + \sum_{|\Delta_b|=2} z^b \partial_b$$

Then $E\Phi^V_{qc} = (3 - dim_{\mathbb{C}} V)\Phi^V$.

I'll get some part that's generically quadratic. This is an easy exercise, you can check this by looking at how the quantum correction looks.

Now let's look at the F-manifold and Frobenius manifold contexts.

Definition 1. *A local vector field E on an F-manifold is* Euler *if:*

- *(compatibility with the product) $P_E(X, Y) \equiv d_\circ X \circ Y$ where $d_\circ$ is called the weight of E This can be written differently, as $Lie_E(\circ) = d_\circ \circ$.*
- *(compatibility with a compatible flat structure, if it is present) I want $[\mathcal{T}^f_M, E] \subset T^f_M$ with respect to the adjoint action.*
- *(compatibility with metric, if present) I want $Lie_E(g) = Dg$, where D is a constant eigenvalue, and if you don't want to think in terms of Lie derivatives, I can write $E(g(X, Y)) - g([E, X], Y) - g(X, [E, Y]) = Dg(X, Y)$.*

This produces a spectrum. We normalize d_0 to 1, normally, then the spectrum of $-ad\ E$ on $\mathcal{T}^f_M$, then D.

The exercise for you, looking at the expression given for E, check that the compatibilities are satisfied and calculate the spectrum.

As an example, assume you are in the semisimple case so you have vector fields $e_i = \frac{\partial}{\partial x_i}$ where $e_i \circ e_j = \delta_{ij} e_i$. Then you can describe all Euler vector fields, there is a basic one and you can add constants d_0 and c_i to get $d_0 \sum (u_i + c_i) \frac{\partial}{\partial u_i}$.

One problem that is still unsolved, is, what has generically semisimple quantum cohomology. One restrictive contdition comes from this classification. One should only return to quantum cohomology and notice that beside this, you can replace this E with similar things that take into account the Hodge structure. If you replace a by p or by q, you get two different Euler fields if and only if [unintelligible]. The two commuting Euler vector fields is impossible in the semisimple case. If $H^{p,q} \neq 0$ then $p = q$ for a semisimple Frobenius manifold. At first this was true just for the even dimensions, but it's also true for the odd case.

So semisimplicity can hold only when all cohomology is of type $H^{p,p}$. I guess I'll pass to the Saito structure next time.