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I'm continuing the introduction to quantum cohomology. As a recollection, I was starting with a preliminary incomplete (imprecise) but intuitive definition. I feel that without such an intuition one cannot really work with this stuff. We start with V a projective smooth algebraic variety. Consider its space $H^*(V)$ of cohomology with complex coefficients. One variable will be an integral class β modulo torsion in two-dimensional homology, which we use to classify algebraic curves lying in V. If you fix a certain number of degrees, the family will be finite dimenisional. Not any cycle can be one of these, there is an effectivitiy condition. You could choose a projective embedding and then make a choice based on this arbitrary embedding, but choose all possible embeddings (ample divisors), so they should be nonnegative with respect to any ample divisor. So $(\beta, [D]) \geq 0$ for any ample divisor D.

Then I will have a variable number $n \geq 0$, and the basic Gromov-Witten invariant $I^V_{0,n,\beta} : H^*(V)^{\otimes n} \to \mathbb{C}$, where 0 reminds you that we're only considering genus zero curves. I have written you an intuitive description of these numbers. It suffices to describe these numbers $I^V_{0n\beta}$ on a basis Δ_a of cohomology modulo torsion, represented by cycles D_a. To define the full story, $I^V_{0n\beta}(\Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n})$ I must know this for every choice of a_j. This will be the *virtual* number of rational curves C of class β with n marked points $x_1, \ldots, x_n$, pairwise distinct (and for now nonsingular but we'll talk about that in more detail later) which intersect D_{a_i} and so that the intersection contains x_i, then algebraic geometry says that the number of such curves is virtually finite. If it is infinite then we put 0.

I will consider not just curves lying in V but maps of curves into V. Suppose you want to define the fundamental group. Your first idea is closed loops at V starting and ending at a point. But we know you should consider parameterized loops. You should not imagine the curves sitting in V but mapping to V. The map on its image might have finite degree, but then the class should be multiplied by the degree to get the correct β. There will be much more detail later on.

Now I will use the intuitive definition in the following way. I will produce a list of properties that such numbers can be expected to have. From this intuitive definition, I will try to make each statement of this list. I will call them "axioms." Then, when this is done, I will construct from $I^V_{0n\beta}$ the potential of a formal Frobenius manifold.

On the other hand, of course, in 1994 it was a challenge to construct these numbers with these properties, and when it was done, it turned out that it gives you much more than just these numbers, and this is a big motivic picture in algebraic geometry. Then there arises an interesting challenge. So far as we know it is not defined in terms of axioms. It might happen that there are other properties, but we do not know all the properties. We should either prove some desirable properties, or we can say that this incomplete information gives us more than what we started from in the beginning. This is the source of a dozen papers, reconstruction theorems letting us construct all of the data from some of it.

Now I pass to the list of properties. For each one I will have an intuitive explanation, and then say how it is reflected in the properties of the potential Φ.

0.1. **Axioms.**

(1) Effectivity: $I^V_{0n\beta} = 0$ unless β is in the positive lattice B (with respect to the positive divisors). When $\beta = 0$ we should get something particular but I will defer that.

(2) S_n-invariance. So S_n acts simultaneously on $I^V_{0n\beta}$, and we renumber the points x_i that mark our curve. The result should remain the same. It shouldn't matter what numbers are here. It is important because it explains why my Φ should be the characteristic function of commuting variables.

(3) Degeneration. This is nonobvious but the formula is very nice. Fix a representation $\{1, \ldots, n\} = S_1 \sqcup S_2$ with n_1 and n_2 elements. Then consider decompositions of β into $\beta_1 + \beta_2$ (with $\beta_i \in B$)

$$I_{0n\beta}(\gamma_1, \ldots, \gamma_n) = \sum (I^V_{0(n_1+1)\beta_1} \otimes I^V_{0(n_2+1)\beta_2})(\bigotimes_{S_1} \gamma_j) \otimes \Delta \otimes (\bigotimes_{S_2} \gamma_k)$$

where Δ is the class of the diagonal in $V \times V$ which sits in $H^*(V)^{\otimes 2}$, and in a basis, this is $\sum \Delta_k \otimes \Delta_\ell g^{k\ell}$.

We have met something like this in the Kontsevich formula for $\mathbb{P}_2$. I'm considering a rational curve with some market points in different parts, and the idea is that it can degenerate into a curve consisting of two irreducible components. Some part of the marked curves will go to one part, S_1, and some to the other S_2. This intersection point is also a marked point, but these should be in such a way that these marked points coincide.

The class β should remain the same, so βs should be the one and other side. You should know how the components intersect each other. This explains why in the Kontsevich formula there were funny binomial numbers, that comes from this combinatorics. This must be proved, but this shows the geomtery behind things. Now I am explaining why something like that should be true, but this is basically the most important part in showing that the potential should satisfy the associativity conditions. This is pretty formal for the third derivatives of Φ with one index raised, Φ^c_{ab}.

[Question about stability.] I will speak about that in a lot of detail later.

(4) Identity. $I^V_{0n\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes 1)$, where 1 is dual to the fundamental class, this is the same as $I^V_{0(n-1)\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1})$. Intuitively this is clear, there is no condition in this case.

(5) Divisor. $I^V_{0n\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \underbrace{\delta}_{} \text{in } H^2) = (\beta, \delta) I_{0(n-1)\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{n-1})$. I have choices about which is x_1, which is x_2, and so on, and that gives me (β, δ).

What can we conclude from these? That the identity of the usual cohomology is still the identity. From the divisor axiom it follows that, well, in the characteristic function, one argument was appearing in the exponential. From this it will follow that it is reasonable to exponentiate all arguments in the potential (linear dual coordinates to a basis in the classical cohomology) that are in topological codimension 2.

(6) Dimension. I will state it as follows. If $I^V_{0n\beta}(\gamma_i \otimes \cdots \otimes \gamma_n) \neq 0$ then

$$\sum (|gamma_i| - 2) + 2[(\underbrace{K_V}_{\text{canonical class}}, \beta) + 3] = 2 \dim_{\mathbb{C}} V.$$

This is difficult to explain intuitively, but it's very useful, because it restricts those monomials that can appar in Φ. If you endow your flat coordinates or the exponentiated coordinates with weights, then with respect to these weights your Φ is homogeneous.

Let's look at an important case. Knowing the identity and divisor axioms, I can assume all γ_i are more than 2. Let's consider the case when $K_V = 0$, Abelian varieties or Calabi-Yau varieties or something like this, we get $\sum(>0) + 6 = 2\dim V$. Consider the case when the dimension is 3, so 3-dimensional Calabi-Yau's. Then the arguments are empty, that's the only time when this can be nonzero. Only then, $I^V_{0,0,\beta}(1)$, you may expect that things are nonempty for these values. But β remains totally restricted, since it cancels with K_V. In the Calabi-Yau thing, you are only interested in the degrees of β in a fixed embedding. For them, a fantastic mirror formula was discovered by physicists, Calabi, and then proven in some sense by Givental. So there should be this number of rational curves. Whether this virtual count is the actual algebraic count, that's unsolved. There may be continuous families of β for some Calabi-Yaus but those are not sufficiently generic. To say that for sufficiently generic Calabi-Yau there are this many curves, that's not yet proved. So you cannot avoid deforming. So for $\mathbb{P}^r$ you can understand things in algebraic geometric terms.

(7) "Mapping to a point" ($\beta = 0$). This is not intuitive, it depends on an understanding of what it means, virtual. In this geometry, $\beta = 0$ may not be trivial.

Later on I will explore in more detail the case of projective space. It's essentially the case when associativity gives you all the numbers by a recursive formula. I'll give as a central example quantum cohomology of $\mathbb{P}^r$. This will be indicative of what we can expect from the mirror picture.

Example 1. *Let $H = H^*(\mathbb{P}^r, \mathbb{C}) = \sum_{a=0}^{r} \mathbb{C}\Delta_a$, where Δ_a is a class of codimension a in $\mathbb{P}^r$, and this produces coordinates x_a, and*

$$\Phi^{\mathbb{P}^r}(x) = \frac{1}{6} \sum_{a_1+a_2+a_3=r} x_{a_1}x_{a_2}x_{a_3} + \underbrace{\sum_{d=1}^{\infty} \Phi_d(x_2, \dots, x_r) e^{dx_1}}_{\text{“quantum correction”}}$$

where

$$\Phi_d(x_2, \dots, x_r) = \sum_{n=2}^{\infty} \sum_{a_1+\cdots+a_n=r(d+1)+d-3+n} I(d, a_1, \dots, a_n) \frac{x_{a_1}\cdots x_{a_n}}{n!}$$

where $I^{\mathbb{P}^r}_{0n\beta} = d\ell(\Delta_{a_1} \otimes \cdots \otimes \Delta_{a_n})$, where ℓ is the class of a line.

Theorem 1. *Assuming the existence of $\Phi^{\mathbb{P}^r}$, it is uniquely reconstructed from $I(1, r, r) = 1$.*

Even the existence can be proven combinatorially. This takes a lot of not really illuminating computations. Exactly one power series with these properties and this coefficient exists.

You now have a formal Frobenius manifold which is the quantum cohomology of $\mathbb{P}^r$. Any questions?