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Okay, um, some recollection.

(1) First of all, we had a notion of F-manifold. We take any manifold M with structure sheaf $\mathscr{O}_M$ and tangent sheaf $\mathscr{T}_M$. You supply $\mathscr{T}_M$ with a commutative associative $\mathscr{O}_M$-linear product $\circ$. Sometimes you want an identity, and this should satisfy the F-identity. This is an F-manifold.
(2) Then there might be a flat structure which consists of a sheaf $\mathscr{T}_M^f$ of flat vector fields whose bracket is zero, which form a local system of linear spaces, which after tensoring with local spaces gives all of $\mathscr{T}_M$, which is compatible with the F-manifold structure if multiplication restricted to flat vector fields is locally $X \circ Y = [X, [Y, C]]$.
(3) You could have a compatible "metric" on flat vector fields with constant values, so $S^2\mathscr{T}_M^f \to \mathbb{C}$, say. This should be nondegenerate, and should have a compatibility with multiplication so that $g(X \circ Y, Z) = g(X, Y \circ Z)$. So I already mean the $\mathscr{O}_M$-bilinear extension to $\mathscr{T}_M$. This total package is called a Frobenius manifold.

In the next few classes I will explain two structures giving rise to these, but I have a few other things to say first.

Theorem 1. *If I have $(\mathscr{T}_M, \circ)$ where $\circ$ is $\mathscr{O}_M$-bilinear, commutative, and associative, along with the flat structure with compatibility but not necessarily the F-identity, then the F-identity holds automatically.*

Proof. Calculation, what does it involve? First of all, the F-identity. I will not repeat it once more, it involves a sum of three terms, each of which involves the Poisson tensor $P_X(Z, W)$, which is itself the sum of three terms involving multiplication and the bracket. So there are nine terms involving bracket and product. If the flat structure and compatibility are given, then I can replace each product with a term involving commutators. So I'll get nine terms involving four vector fields (plus C) and a lot of brackets between them. It suffices to check the F-identity only for flat vector fields. You can see, sometimes, if a bracket involves Y and C you would keep it, but if it were, say, Y and X, those are zero. Then in simplifies to five monomials involving brackets involving the five arguments. You want to prove that this is zero. What do you know? What is actually zero? You know that Jacobi is zero. You also know implicitly that $X \circ Y$ is associative. I didn't invent any nice way to check this, I just put myself in local coordinates, but I think someone can do better. It's simple-minded but the fact is nice and useful. $\square$

Now let me sketch two constructions of F-manifolds and Frobenius manifolds. We will see the machinery in the construction are as unlike as one can imagine. When one says that there is an isomorphism between things on the left hand side and the right hand side, that's saying something about the mirror, because these are very much different.

So one example is quantum cohomology $H^*_{\text{quant}}(V)$, where V is a smooth projective irreducible manifold.

On the other side the initial data will be Saito's structure, which I will describe a little later on.

I will only get formal series on the left hand side, but these are equivalent to the ones on the right hand side, their completions, and these are local or even global.

It's interesting to address these questions of locality [more detailed description of problem missed].

So for quantum cohomology, M will be a formal completion of the linear $\mathbb{C}$-space $H^*(V) = H^*(V, \mathbb{C})$. The local functions $\hat{\mathscr{O}}_M$ will be a ring of formal series in linear coordinates on $H^*(V)$, except for H^2, where we will use exponentials of linear coordinates. This is part of a more general phenomenon. We should think of the grading, the topological grading and Hodge grading as interacting with the Frobenius structure.

So $\mathscr{T}_M$ is replaced by the initial linear space $\underbrace{H^*(V)}_{\text{flat vector fields}} \otimes \hat{O}_M$.

The multiplication is, well, let me say $H^*(V) = \bigoplus \mathbb{C}\Delta_a$ where Δ_a are dual classes of some cycles x_a on V, where $|\Delta_a| \in H^{|\Delta_a|}(V)$. I'm proceeding as if all cycles are even dimensional, so that I can suppress signs. It's formal to move to supergeometry but I will pretend that things are even dimensional to make things easier. Imagine that you are considering only the even part of cohomology. Then $\Delta_a \sim \frac{\partial}{\partial x_a}$. So the multiplication is $\Delta_a \circ \Delta_b = \sum \phi_{ab}^c \Delta_c$, where ϕ is a very concrete and difficult to define formal series whose coefficients count rational curves of V with incidence conditions. The index c is raised with the intersection pairing.

This is part of a much larger story, Gromov-Witten invariants, also things that are for higher genus. These equations for ϕ follow from subtle algebraic geometry which says how you can count things as they degenerate.

Okay. This is only a sketch. I've finished temporarily. On the other side, in Saito's story, the input is this: M is a "space of deformations." In the most general case it's very unsophisticated. It should be a diagram in which the first part is a surjection of complex analytic spaces $N \to M$. It is actually a submersion, which means essentially that everywhere locally you have a system of coordinates (z_a, t_b) on N and the projection forgets z and becomes a system of local coordinates (t_b). So it's locally a product of complex analytic manifolds.

Now we will also need a function $F = F(z,t)$ from N to $\mathbb{C}$. Now, knowing this function, one can produce in N the subspace C of its critical points. C is a complex analytic subspace which is locally given by equations $\frac{\partial F}{\partial z_a} = 0$. So I'm considering the t as parameters. So $F(t,z)$ I should think of as $F_t(z)$. I should have a closed embedding $C \hookrightarrow N$ so that gives me the projection $p_C : C \to M$.

If I have all of this, I have the following. I can consider local functions on C as a sheaf on M. More precisely, if I have an open subset U of M then $P(U, p_{C*}(\mathscr{O}_C)) = P(p_C^{-1}(U), \mathscr{O}_C)$.

Now I have a map $s : \mathscr{T}_M \to p_{c*}(\mathscr{O}_C)$. So lift X to $\bar{X}$ on N nonuniquely. Apply this to F and then restrict the result on C. So you get $X \mapsto \bar{X}F|_C$. Of course, $\bar{X}$ is not unique. But $\bar{X}' - \bar{X}''$

is a linear coefficient of partials with respect to z. So when I restrict to C the indeterminacy vanishes. It also shows that it's sufficient to consider local things.

Now, this is the basic assumption, #1, I want to assume that the initial diagram is such that s is an isomorphism of $\mathscr{O}_M$-modules. In particular the pushdown is locally free, the rank is the dimension of M, and so on.

There are several different cases in the literature where you assume something about F and prove it's an isomorphism. One is, start with $F(z,0)$. Look at the singularities of this, the ramification points, the critical points of this story, and then you have an isolated singularity. You start with just one fiber with an isolated singularity. Then it is well-known the construction of the versal (sometimes universal) unfolding of the singularity. The simplest example is when the function is z^{n+1}. Then from this fiber, you get the isomorphism condition.

Now, if this assumption is satisfied, then you get automatically an associative commutative multiplication on $\mathscr{T}_M$, because $\mathscr{O}_C$ is a ring of algebras. Then there's the question, is there a compatible flat structure or a metric? Doing this was an important achievement of Saito. This was the most difficult part.

You will see that inputs are different, and the order of difficulty is opposite. In quantum cohomology, the multiplication is very difficult, and the flat structure and metric come from the start, whereas in the case of Saito's structures, the multiplication comes from the start, and moreover it's extremely natural, and the flat question is the difficult part. When you establish an isomorphism, they come in a quite unexpected way. One doesn't understand very well why you get isomorphisms.

This was a sketch. Now I give some more details.

There is a paradoxical situation that was hard to get accustomed to. So $H^*(P^z) = \mathbb{C}[x]/x^{z+1}$. It looks like the most natural candidate would be the singularity $x^{r+1} = 0$ and then unfold it. There is a lot of literature by people who tried to exploit this idea. Often the cohomology ring is a ring modulo derivatives of something. So it would be most natural to see this. It never works. I still think we do not understand something important, but we always get troubles with the flat structures from one side to the other. I'll show you what seems to be happening. There's something we don't understand.

So, now I will deal with the left-hand side and start with Gromov-Witten invariants, which from our viewpoints are coefficients of the potential whose third derivatives produce the multiplication in quantum cohomology. We'll start with a preliminary definition—for a complete definition, we would need a lot of technique that we have no time for.

Definition 1. *(preliminary and incomplete)*
Let V be a smooth projective variety, manifold, then, probably it's nice to choose a basis (Δ_a) of $H^*(V,\mathbb{Z})_{tors}$ or $H^{even}(V,\mathbb{Z})_{tors}$. I will try to keep notation very stable. So β will always denote an element of $H_2(V,\mathbb{Z})/tors$ which is part of a lattice in the space dual to $H^2(V,\mathbb{C})$. Each rational curve has intersection indexes with 2-cycles. So this supports classes of rational curves (not necessarily rational as this moves to higher genus) in V as represented by intersection indices with codimension 2 classes.

Then I will choose a number $n \geq 0$ and then my standard notation for the Gromov Witten invariant will be, if V can change I put $I^V_{0,n,\beta}$, where V is my space, 0 is the genus, n counts *an intersection number, and β is as defined. I will think of this as being a map $H^*(V)^{\otimes n} \to \mathbb{C}[often\ \mathbb{Z}\ or \mathbb{Q}]$.*

This is a GW-form or correlation function, an n-point correlation function in physics speak. A GW-number is one of the coefficients of one of these forms.

So I could consider $I^V_{0,n,\beta}(\Delta_{a_1}, \cdots, \Delta_{a_n}$.

The idea is that this should be roughly the number of rational curves in the homology class of β and intersecting some sufficiently generically cycles D_{a_i} representing Δ_{a_i}, whenever this number is finite, (0 otherwise).

There are not very sophisticated genericity conditions for choosing D_a's, but something that's very hidden as a genericity conditon refers to deforming, the possibility to deforming V itself.

For example, take $\mathbb{P}^2$ with three blown up points. These lie on a line. But the naive count gives an incorrect answer for V_2 but correct for V_1. This is why there are two disjoint communities. In symplectic geometry there is freedom for manipulation. But here you cannot even deform V to make it correct. With too much genericity, algebraic geometry turns out to be too rigid.