YURI MANIN

GABRIEL C. DRUMMOND-COLE

We were talking about F-manifolds and I was comparing them to Poisson geometry. You have two basic sheaves, functions and derivations, which form a commutative algebra and a Lie algebra. We can just by hand decide that we want functions to also have a bracket or vector fields to have a commutative associative multiplication. In the first case we will get a Poisson structure if we have an additional identity; in the second we will get an F-manifold structure if we have a more complicated identity.

At the end of last class, I announced a theorem which was a precise relation between these two, which I will now repeat and prove.

Recall: if you have M a manifold, C^∞, analytic, or things like that, you have the tangent sheaf TM, that means the space or bundle with T square, or T^*M the dual cotangent bundle. The classical fact is that T^*M has a canonical symplectic structure. It can be defined in a noninvariant way, or we can choose local coordinates q_i on M and produce the differential of them, these are sections of T^*M; denote these by p_i. Then in these coordinates, if you take functions f and g, then their bracket $\{f, g\}$ is $\sum \partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f$, up to sign. I studied this in the course of mechanics. They say, just take the position space and momentum space, and you write down $\partial_t F = \{F, H\}$, and this gives a system of equations. I found this extremely mysterious, and only very gradually do I start to feel I understand something about it. If you have a not very sure feeling that you have seen it somewhere, it is very important. This was as mysterious to me as astronomy. The question was something about the movement of the visible moon, and I was so confused that I told my professor that for me the moon and sun were the darkest planets. You need time and you need hard thinking.

You want to pass to F-manifolds, so we'll add a commutative multiplication $\circ$ from $S^2_{\mathscr{O}_M}\mathscr{T}_M \to \mathscr{T}_M$ along with maybe an identity e. Sometimes this will satisfy the F identity and sometimes not. Let's say what it means. This is the same thing as giving a map of algebras $Symm^*_{\mathscr{O}_M}(\mathscr{T}_M) \to (\mathscr{T}_M, \circ, e)_{\mathscr{O}_M}$. How do you define this? You send $f \in \mathscr{O}_M$ to fe and you send $\mathscr{T}_M$ to itself.

Such a homomorphism defines its kernel $J(\circ)$. The second remark is that that the sheaf $Symm^*_{\mathscr{O}_M}(\mathscr{T}_M)$ are not all functions on T^*M, they are polynomial with respect to fibers. From this formula it follows that this sheaf of rings is stable with respect to the canonical bracket $\{\ ,\ \}$ on T^*M.

Now we can formulate the theorem:

Theorem 1. *$\circ$ satisfies the F-identity if and only if $\{J(\circ), J(\circ)\} \subset J(\circ)$.*

Proof. I will divide this into several steps, skipping the computational details. Each step is a small lemma. Generally, if you have a Poisson manifold J is an ideal consisting of some functions on it, to check that J is closed under the bracket is the same as checking that there is a system of generators so that the bracket of them is contained in J.

Step two, we want to apply this to $J(\circ)$ on T^*M, so we want generators . Choose generators. From what I have written, it's pretty clear that one possible and obvious choice is $e-1 \in Symm_{\mathscr{O}_M}\mathscr{T}_M$ and $X\cdot Y - X\circ Y$. It's pretty easy to see that this is all you need.

The next thing to check is the easy part, that $\{e-1, e-1\} = 0$, that if X and Y are local vector fields then $\{X, Y\}$ is the same as $[X, Y]$. This is slightly more complicated. This will help us to calculate the commutator of $X\cdot Y - X\circ Y$ with other things.

Now there is a direct computation using what you know about Poisson brackets. If you commute $\{e-1, X\cdot Y - X\circ Y\} = [e, X\circ Y] - X\cdot[e, Y] - [e, X]\cdot Y$ and also that $\{X\circ Y - X\cdot Y, Z\circ W - Z\cdot W\}$, and this is the most important part of the calculation, that here you have the F identity up to $J\circ$, where the F identity is $P_{X\circ Y}(Z, W) - X\circ P_Y(Z, W) - Y\circ P_X(Z, W)$. Again, later I will write down, this is three terms, so there are nine terms here. Actually if you studied a little bit of operads, you should think I'm doing something that is related to operads, and that's very fruitful, you can see something about this in Merkulov, and I'll discuss this later.

So the remaining step, we want to prove that the two statements are equivalent. If the F-identity holds for $\circ$, then you have, using all of this, for our generators, all of of the commutators land in J and from the left hand side the right hand side follows.

For the reverse direction, if we assume that the bracket of J with itself is in J, we get that we can project this into the degree 1 part of the symmetric algebra, and we know from the right-hand side that we land in the intersection of the degree one part and J, but in $\mathscr{T}$ there are no pure degree one parts of the relations, so the left hand side must be zero. □

This theorem clarifies the relationship between Poisson and F-structures, and as I mentioned, although from the viewpoint of subspaces, not submanifolds, we generally had the spectral cover $\tilde{M}$ defined from $\mathscr{T}_M, \circ, e$. So impose $J = 0$. You get a closed subspace. It's a subspace of T^*M, and in the space of F-manifolds we call this the spectral cover, and this shows, the closedness shows, that in the generalized sense it's a Lagrangian subvariety of T^*M, but not a Lagrangian submanifold. A spectral cover is a "generalized Lagrangian submanifold."

A lot of things that were known in the geometry of Poisson and symplectic manifolds can be used on this spectral cover, but not everything.

Let me return to quantum cohomology. You have $H^*(V, \mathbb{C})$, and then you have this tensored with "formal functions." In this analog, $H^*(V, \mathbb{C})$ are formal vector fields on M and I have coordinates that are linear in some variables and exponential in others, and so this tensor is something like formal vector fields. This is obviously very special. We should introduce an additional structure, the flat structure to deal with this special case.

I'll be assuming that I have mostly F-manifolds but sometimes manifolds with just a multiplication on the tangent sheaf.

In quantum cohomology I had a kind of completion of TM at one point. In the Kontsevich example I had two coordinates and then one that was as exponential. The flat vector fields here didn't have a cognate in the F-manifold part.

I am missing some pieces. One is [unintelligible]. Another is the intersection form, and finally I have a $\mathbb{Z}$-grading. In general F-manifolds I will have the Euler field. So I'll pass to the more stringent notion of Frobenius manifolds.

Let's talk about flat structure(s) on M. There are three equivalent forms, and it's useful to have all of them in mind.

(1) I can give on M a complete atlas of local coordinates such that, well, if you have an atlas, you have transition functions. If all transition functions are affine with constant coefficients, that's a flat structure. Those local coordinates that form the part of such an atlas are flat local coordinates.
(2) Instead I can consider the subsheaf $\mathcal{T}_M^f$ of vector fields. In terms of the previous definition, they are fields that generated by partial derivatives of one system of flat local coordinates. Invariantly, it is a local system of Lie algebras with trivial bracket so that $\mathcal{T}_M \leftarrow \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{T}_M^f$ is an isomorphism.
(3) The third equivalent definition uses the notion of a connection on the vector bundle. Consider the connection $\nabla : \mathcal{T}_M \to \Omega_M^1 \otimes \mathcal{T}_M$. It defines a covariant derivative of avy vector field. $\nabla(X)$ lands in, it's of the form $\sum df_i \otimes Y_j$, and you know the axiom in this context, $\nabla(fX) = df \otimes X + f \otimes \nabla X$. Of course, in this general definition, you could take any vector bundle, and any local section, but in the specific case that it's TM, this is dual to $\Omega^1(M)$, and you have additional linear algebra so that you can define torsion and curvature. To define a flat structure, I want both of them to be zero. To get to the second definition, we have $\mathcal{T}_M^f = \ker \nabla$.

Now we return to the case where we have a multiplication on $\mathcal{T}$. There will be a restriction.

Definition 1. *Suppose that you have M and $(\mathcal{T}_M, \circ)$. I'll say a flat structure $\mathcal{T}_M^f$ is compatible with $\circ$ if everywhere locally, there is a vector field C in $\mathcal{T}_M$ so that for X and Y flat vector fields, $X \circ Y = [X, [Y, C]]$.*

Some remarks. I'm defining the compatibility of the multiplication and the flat structure. This defines a commutative multiplication of flat vector fields, which then can be extended to a commutative multiplication of vector fields, not just of flat vector fields. Why is this commutative? It's because of the Jacobi identity.

The second remark, what does this formula mean in flat coordinates? If you take an open chart x_i, write C as $\sum C^k \partial_k$. Then make a small calculation to see how you multiply ∂_i and ∂_j, and you will get

$$\partial_i \circ \partial_j = [\partial_i, [\partial_j, \sum C^k \partial_k]] = \sum C_{ij}^k \partial_k$$

where $C_{ij}^k = \partial_i \partial_j C^k$. This is close to quantum multiplication. There I took a third partial derivative, and lifted the last subscript using the intersection form. Here is the best approximation in the absence of such a quadratic form. It's clear I made a step toward quantum multiplication.

The next step if I have such compatible things:

Definition 2. *If I already have $\circ$ and a compatible flat structure $\mathcal{T}_M^f$, then I will say that $S^2\mathcal{T}_M \to \mathcal{O}_M$, a formal metric g is compatible with this story if it can be obtained from its restriction on $\mathcal{T}_M^f$, and on flat vector fields it will land in constants.*

If you write this matrix as $g_{ij}dx_idx_j$ and take for x_i and x_j, then this g_{ij} should be a constant. I also want this to be a metric. I want it to be nondegenerate. I'm looking over $\mathbb{C}$ but at least it should be nondegenerate over each point.

Now we are in a situation which is formally quite close to quantum cohomology. We can now lower the index. It's not all that we want. This is just the first condition of compatibility. I also want compatibility with $\circ$, which is the Frobenius property, that $g(X, Y \circ Z) = g(X \circ Y, Z)$.

This is almost everything we took into account. We have a flat structure, we have a multiplication whose constants are the third derivatives of something with a lifted subscript. I will need to check that the properties I have postulated give me the properties I want.

I want to finish today with a nice surprise. I will state it and probably prove it the next time. The F identity lurks behind. If I do have this structure, it turns out that the F-identity is automatic.

Theorem 2. *If $(\mathscr{T}_M, \circ)$ admits a compatible flat structure, then M together with $(\mathscr{T}_M, \circ)$ is an F-manifold.*

Now only one more remark. In quantum cohomology we have this power series, the potential is a characteristic function counting rational curves, and now in a general framework this corresponds to our vector field C, some vector fields. How unique is it? We want to fix multiplication, so you want to fix all third derivatives, so modulo constant, first, and second derivatives, it is unique. How about C? It's pretty clear that it's almost the same. If you add an arbitrary flat vector field or a flat vector field multiplied by a flat coordinate, the expression will not change. So C is defined modulo flat vector fields and flat vector fields multiplied by a linear expression in flat coordinates.