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Today I'm going to talk about F-manifolds. Recall that I talked last time about how many problems come from counting solutions to equations. Recall that we talked about the Jacobi identity, which says that the number of decompositions of n into four squares is equalto $8\sum_{d|n, 4\nmid d} d$.

The proof is, you produce a characteristic function $q(n)e^{2\pi i z n^2}$, and from the definition, this becomes $(\sum e^{2\pi i z m^2})^4$, and the thing inisde the parentheses is a version of a classical theta function, which satisfies something like $\theta(z+1) = \theta(z)$ and $\theta(-\frac{1}{z}) = \cdots$ This embeds in the theory of fractional linear transformations of the upper half plane, which is the same as the theory of Riemannian surfaces (of highe enough genus). Once we understand that, you can use elementary transformations to show that the other function satisfies the same properties.

This is a very important strategy for deailing with this sort of question. It's nontrivial, and one of the highlights of quantum cohomology was understanding that this can be applied to very concrete problems in algebraic variety. In the simplest case you count rational curves of a given degree. You put controlled incidence conditions asking for the curves passing through several points. Before physicists came, nobody figured out how to make the characteristic function.

There are famous classes, Taylor series, Fourier series, Dirichlet series, you want to land in something that has a rich theory itself. What happened more concretely in the example I gave? This was Maxim Kontsevich's example. We defined $N(d)$ to be the number of rational curves of degree d in $\mathbb{P}^2$ passing through $3d-1$ points.

How do you choose a generating function? We take

$$\Phi^{\mathbb{P}^2}(x,y,z) = \frac{1}{2}(xy^2 + x^2 z) + \sum_{d=1}^{\infty} N(d)\frac{z^{3d-1}}{(3d-1)!}e^{dy}$$

The transition itself here is physics. But as soon as you have invented this, what you do, you produce from Φ a commutative associative algebra over, well, it depends on coefficients. You can take any field of characteristic zero, let's say $\mathbb{Q}$ for now, it will be an algebra over, a module over $\mathbb{Q}[[x, e^y, z]] \otimes_{\mathbb{Q}} H^*(\mathbb{P}^2, \mathbb{Q})$. Then we define a multiplication, a deformation of the multiplication on this tensor product. On the right you have three canonical generators Δ_0, Δ_1, and Δ_2, where Δ_a is the dual class of the projective subspace of codimension a. Now imagine a generic element is a linear combination of these. You defom the multiplication by taking as our multiplication constants

$$\Delta_a * \Delta_b = \sum_c \Phi_{ab}^c \Delta_c$$

where the index c is lifted using the intersection form. This is one of Einstein's interesting discoveries, when you sum over repeated indices you can skip them. I will normally not skip them. This is a joke.

So it's obvious that this is commutative, but it's not at all obvious that this is associative. Associativity conditions are also called Witten-Dijkgraf-Verlinde-Verlinde equations, this is a big system of PDEs for Φ, they are quadratic and of degree 3. You write it

$$(\Delta_a * \Delta_b) * \Delta_c = \Delta_a * (\Delta_b * \Delta_c)$$

and you write this down, it's highly nonlinear, something that has not been seen in PDE theory before. It's a deformation because if you look just at the classical part, killing the other piece, you get the regular multiplication.

How should I generalize this to pass to the subject matter? Replace $H^*(\mathbb{P}^2, \mathbb{Q})$ by a manifold M which you can understand in any category you are happy about. Maybe sometimes schemes, but pretty soon you will have to integrate and you can't integrate there. For quantum cohomology it should be a supermanifold. I will imagine always it's just a manifold.

So then you replace $\mathbb{Q}[[x, e^y, z]]$ with a ring (or sheaf) of formal functions on M, and here on the right, it's not just a manifold, it's a linear manifold, so we can imagine that these are just vector fields on M, so replace $H^*(\mathbb{P}^2, \mathbb{Q})$ with the space of vector fields so that Δ_a corresponds to derivation with respect to x_a.

This corresponds to a passage to something, we don't know yet exactly what this is. Explanations of why we have chosen this form of generating function comes later on when we see we have landed in a very interesting class. So now I'll describe the geometric thing we have built forgetting the algebro-geometric background of where it comes from. This is the logic for what I'll be doing for some time and for some time you may forget about algebraic geometry.

Preliminarily, we will start by choosing a manifold M (maybe C^∞, analytic, or whatever), and then the sheaf $\mathscr{O}_M$ of local functions, and then the sheaf $\mathscr{T}_M$ of vector fields or local derivations, and you have because this is a manifold, linear maps satisfying the Leibniz rule, if I want manifolds, $\mathscr{T}_M$ should be a locally free sheaf, $\mathscr{T}_M \approx \mathscr{O}_M^n$ where $n = \dim\ M$. Now, $\mathscr{O}_M$ has its own natural product, these are commutative associative algebras, and $\mathscr{T}_M$ is a sheaf of Lie algebras. and these are related by several identities, the Leibniz identity, and you may consider as an exercise, $\mathscr{O}_M \times \mathscr{T}_M$ and, you have maps $\mathscr{T}_M \times \mathscr{T}_M \to \mathscr{T}_M$, you have a map $\mathscr{O}_M \times \mathscr{O}_M \to \mathscr{O}_M$, you have multiplication $\mathscr{O}_M \times \mathscr{T}_M \to \mathscr{T}_M$, and you can apply $\mathscr{T}_M \times \mathscr{O}_M \to \mathscr{O}_M$. You can write down all the identities you know, they're all classic.

There is a classical idea in geometry, which says that another natural structure, a Poisson structure, well, if you add a bracket $\{,\}$ to $\mathscr{O}$, when we consider quantum cohomology, we are led to an analogous structure, for an F-structure, we'll say that $\mathscr{T}_M$ will have an additional commutative associative structure $\circ$. We will see how these are similar different, and related. Poisson manifolds are classical and F-manifolds are new and not so well-studied.

Now, to write down the main axiom, well, in Poisson, suppose you have a commutative ring A with a multiplication $\cdot$ and bracket $\{,\}$. Suppose everything is an algebra and linear over a field and so on. You can produce the Poisson tensor. If $x, z, w \in A$, the Poisson tensor is $\{x, z \cdot w\} - \{x, z\} \cdot w - z \cdot \{x, w\}$. In the super case you have to introduce the signs. You produce if you have such a structure, you have a three-linear function of A. The axiom for a Poisson structure is quite simple: you take functions, multiplications, and the Poisson bracket, and the Poisson tensor is identically zero, $P_x(z, w) = 0$. The main axiom for the F-structure, the F-identity, is more sophisticated, you work in $\mathscr{T}_M$ which has the new $\circ$ multiplication as well as the classical bracket [,], and then it'll be four linear, the tensor is

$$P_{X\circ Y}(Z,W) - X\circ P_Y(Z,W) - Y\circ P_X(Z,W) = 0$$

and let me remind you, $\circ$ is a commutative associative algebra over formal functions. In most applications it has an identity e, and sometimes I don't need identity, and then you have this identity.

Let's play just a little bit with this. In the Poisson case, one can interpret this $P_X(Z,W) = 0$, if I consider $f \in \mathscr{O}_M$, and then $\mathscr{O}_M \to \mathscr{O}_M$ the map that sends g to $\{f,g\}$, this defines a vector field X_f, that's what the identity says, if $P_X(Z,W) = 0$.

So adding the bracket, I can produce from any function a vector field. So in the F-manifold situation it should be vice-versa, any vector field should become a local function, not on M but on a very important object that I will call the spectral cover $\tilde{M}$ over M. What is $\tilde{M}$? I have a sheaf of commutative rings on M, free modules of finite type over functions. So in algebraic geometry, we'd take the spectrum of this ring. You'd get a projection. Later I will define $\tilde{M}$ in more detail, but it's essentially, a vector field on M becomes a local function on $\tilde{M}$.

A couple of easy exercises:

Exercise 1. *Consider any commutative associative multiplication over $\mathscr{O}_M$ on $\mathscr{T}_M$, then we can produce the left hand side of the F-identity. This depends linearly on the four arguments so is a tensor.*

So to check that this is zero, you can check it on a basis.

Exercise 2. *Assume that there is a local coordinate system u_a so that considering $\partial_a = \frac{\partial}{\partial u_a}$, we define $\partial_a \circ \partial_b := \delta_{ab}\partial_a$, that it satisfies the F-identity, and this has an identity, $e = \sum \partial_a$.*

Using the first exercise this is easy. Then this will allow me to compare the Poisson structure and the F-structure. In the Poisson structure, there is the maximally degenerate structure, the symplectic structure. You can characterize these so that there exists a local coordinate system $\{q_1, \dots, q_n; p_1, \dots, p_n\}$ so that $\{p_i, q_j\} = \delta_{ij}$ and of course they commute among themselves $\{p_i, p_j\} = 0 = \{q_i, q_j\}$. In the F-manifold side, we will call them semisimple. It is clear that the formulas defining them are relatively similar. So the symplectic manifolds should be compared to semisimple F-manifolds, and these are "canonical coordinates."

There is an important difference. Semisimple F-manifolds are much more rigid. You are interested in symplectic local automorphisms, this is an infinite group. The case in the semisimple F-manifold side is finite dimensional and much simpler, you can change $u_i' = u_{\sigma(i)} + c_i$. So it's a much more rigid structure. This is not for any F-manifold, just the semisimple ones.

A useful way to look at this story is this. Suppose you have your manifold M and at each space you have the tangent space T_xM, and you get a multiplication on T_xM, call it $\circ_x$, and this is a finite dimensional algebra over the ground field. Say the ground field is closed, say $\mathbb{C}$, you know that the structure of such algebras, every one is the direct sum of local algebras. The semisimple case is, none are nilpotent and all of the local algebra summands are $\mathbb{C}$. Then if this holds in a neighborhood, you can organize these into vector fields that commute, and so the case of semisimplicity of an F-manifold corresponds to semisimple algebras at every point.

We know how to reduce the problem of classification in general to the classification of nilpotent algebras, but this has an interesting structure, these can be deformed (usually coming with a space of deformations). The size of this space depond on the kind of algebra, For commutative algebras it's not so difficult but for associative algebras it's harder.

For F-manifolds, if you have semisimplicity somewhere, you may think you have a a boundary where semisimplicity fails. It can fail in a nonobvious way. Up to now there is no good nontrivial theory of this story. We don't, we can't do it in the generality that we would like.

The only case that has been classified, a beautiful theorem by Claus Hertling, relates to deformations of singularities, and this assumes that $\tilde{M}$ is smooth itself. Then there is a beautiful classification theorem. If $\tilde{M}$ has singularities, or worse, nilpotence, then the classification is difficult and not well-understood.

So I've been comparing Poisson structures and F-structures. I'll formulate and prove a theorem that gives a direct relation between Poisson manifolds and F-manifolds.

So first of all, I have the following classical construction. The point is, if I have an arbitrary manifold M, you produce the cotangent space, a point of M plus a linear functional on the tangent space. This has a canonical symplectic structure. You take local coordinates on M, $q_1, \ldots, q_n$, and then produce p_i which are dual to, well, pointwise we have a cotangent space at each point. There is a basis in tangent in tangent spaces whose basis is $\frac{\partial}{\partial q_a}$. Take p_a dual to this. There is a theorem, if you define this in this way, it turns out not to depend on the choice of q_i.

The second remark is this. Now assume that you have one of the usual $\circ$ multiplications on $\mathscr{T}_M$, but you don't ask for it to satisfy the F-identity. Equivalently, consider the space of polynomials $Symm_{\mathscr{O}_M} \mathscr{T}_M$. We have the multiplication $\cdot$. This extends the multiplication of $\mathscr{O}$. Then there is an $\mathscr{O}_M$-algebra homomorphism to $(\mathscr{T}_M, \circ)$.

The third remark, is, what is this algebra from the viewpoint of $\mathscr{T}^*M$, it's the algebra of functions on M, and along each fiber they are polynomial. Each fiber is a linear space so you know what a polynomial is, so this is the sheaf of functions on T^*M that are fiberwise polynomial.

The fourth point. We have this homomorphism, denote it α, it deponds on the chosen multiplication $\circ$, and consider the kernel of $\alpha(\circ)$. Call this $\mathcal{J}$; it also depends on the chosen multiplication. So $\mathcal{J}$ are functions of some sort on T^*M, but we can take the bracket of these with themselves: $\{\mathcal{J}(\circ), \mathcal{J}(\circ)\}$.

Theorem 1. *The F-identity is equivalent to the fact that* $\{\mathcal{J}(\circ), \mathcal{J}(\circ)\} \subset \mathcal{J}(\circ)$.

This is a very nice theorem that shows you that when you embed the spectral cover $\tilde{M}$ of an F-manifold M in the cotangent bundle T^*M, this embedding is given by equations from $\mathcal{J}$, and you don't generally get this to be a submanifold, but only, say, in the analytic category, an analytic subspace, there can be nilpotence, singularities, so on. If it were a manifold it would be a Lagrangian manifold in T^*M. It's Lagrangian in a generalized sense. The equations that define it are closed with respect to the Poisson bracket. One can construct examples where in the presence of nilpotence or other bad things, but if you forget about nilpotence, [unintelligible].

This tells you you can put the F-manifold theory inside Poisson manifolds, say that you need to consider just Lagrangian manifolds, also Lagrangiang subspaces.