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In the remaining lectures I want to give a very broad overview for Gromov-Witten invariants and to stress that things like that cannot be discovered or stated or studied outside of the realm of post-Grothendieckian algebraic geometry. You must think in terms of functors and stacks and so on. I want to not really explain what the things are but show you how they appear in each stage of Gromov-Witten invariants. You need something at each step, focusing your attention on what is necessary to go on to something else. So it's also some propaganda for modernizing and going even to postmodern algebraic geometry.

1. General Gromov-Witten invariants

My final goal is to show you the way to the following statement: Any motive of smooth projective variety over $\mathbb{C}$ (usually one needs at the present level of knowledge to put a lower bound on characteristic working in characteristic p) is canonically an algebra over the modular operad of motives of (Deligne-Mumford stacks) $\overline{\mathcal{M}}_{g,n}$ (which will be smooth for $n \gg g$. These are specific, they have been studied for many years, but the role in understanding the whole is only understood pretty late, and they are rigid, they cannot be deformed inside algebraic geometry. This is a particular case conjectured by Kapranov.

Up to a pretty recent time, rigidity appeared only in things like simple Cartan Lie algebras, it was discovered that there were quantum groups. About rigid algebraic manifolds, not so much was known. Rigidity was something that seemingly, according to the conjecture by Kapranov, he thought, start with something algebro-geometric, a fixed manifold, probably you should ask that it has not too many automorphisms. If it's not rigid, then consider the whole family of deformations. Look at this total family, see if it's rigid, if it's not, deform it once more. His conjecture is that if you start with a manifold of dimension d, after d steps you get a rigid object. You start with a curve, you deform it, you get a moduli space $\bar{M}_g$ of curves. Since the dimension was 1, you get a rigid object. This was proved by Haiking. If you start with a surface, sufficiently good, and deform it, you will get something that you do not expect to be rigid, and then the deformations of that is rigid again. It's an interesting story, it says that [unintelligible], there are extremely interesting rigid objects in algebraic geometry. Kroenecker said that God created integers and everything else is our work. I would say that God has created rigid objects and everything else sits inside. We know astonishingly little about rigid things.

The first objects we get interact in an extremely strong way with anything. So these objects form an operad and this operad acts on anything at all. The motive you should think of as any cohomology at all.

Reconstruction tells us about concretely describing this. Sometimes the genus zero part can be reconstructed, you should know it for all the symmetric powers. Sometimes the Frobenius manifold itself can be reconstructed with a finite number of integers or rationals. This happens if the cohomology of V is generated by divisors. There are other ways of reconstructing the whole information coming from part of it.

Think of this as an operad, not a ring, because these will act on multiples, not just on one thing.

The sort of mid-level objects encoding this kind of structure are, is a generalization of the Gromov-Witten numbers. It is a set $I^V_{g,n\beta} : H^*(V)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$ where g is the genus of curves by which you are analyzing the geometry, n is the number of marked points, and β is a class in $H^2(V,\mathbb{Z})$ modulo torsion, and $\overline{\mathcal{M}}_{g,n}$ will be the moduli space of a complete algebraic curve and n points marked upon it. In the more or less accepted version, this is the moduli space of stable curves, but one additional circumstance that becomes more and more important is that the condition of stability itself is a variable. So for example, even in genus zero, you might expect to get other invariant sif you change the stability conditions. The same happens in arbitrary genus.

Intuitively, given cohomology classes in $H^*(V)$, $I^V_{g,n,\beta}(\gamma_1\otimes\cdots\otimes\gamma_n)$ is (with a lot of qualifications), if γ is the class of the cycle Γ in V, geometric singular cycles, should be the cohomology class of the cycle in $\overline{\mathcal{M}}_{g,n}$ of those curves for which there exists a map f (stable which I will talk about later) from C to V so that $[f(C)] = \beta$ and $f(x_i) \in \Gamma_i$.

The old Gromov-Witten invariants that I have been considering are essentially, count only the part of $I^V_{g,n,\beta}$ where the dimension matches up and we can count. Everything should be put in general position. So $\langle I^V_{0,n,\beta}\rangle$ was the zero-dimensional part, essentially a number. What is specific about this case is the encoding of this infinite set of data. If it's a sequence of numbers, so it's probably good to encode them as the Taylor coefficients of a series. We got the structure of Frobenius manifolds. We don't know how to encode the whole picture. It would be interesting to know this for higher genus. It's unclear but should be necessary for the mirror phenomenon.

Now, this is the mid-level object, it is less than the full picture but more than numbers, and this part allows a pretty compact intuitive description. Now a little more about how we can proceed to construct it. We cannot actually do this without constructing the whole picture, but once you see what we're doing, you can go downstairs and upstairs pretty easily.

The central object to be constructed is a certain diagram consisting of the following data: We had already implicitly arbitrary powers of V and I had already mentioned the moduli spaces $\overline{\mathcal{M}}_{g,n}$, but you need an intermediary $\overline{\mathcal{M}}_{g,n}(V,\beta)$ (this is one standard notation) parameterizing maps $C \to V$ of class β. It should be a moduli space. You should understand, β is an important variable, but it does a primitive thing. In the algebro-geometric setting, if C is S^1 with marked points, then of course, you can fix the class, conjugacy class and this will be the role of β, a discrete invariant that allows you to split the infinite dimensional space into finite dimensional pieces. If I have a thing like that, I can forget f and produce the curve in $\overline{\mathcal{M}}_{g,n}$, this is not quite correct, you must also stabilize, which involves concretely the stability condition, and can also forget C and then you get $f(x_i)$ in V, so you get an element in V^n. So you have the evaluation morphism and forgetting the map and stabilization. I have invoked this object implicitly telling you about the intuitive meaning of this.

The algebraic geometry to define and prove the existence of these objects as an algebraic scheme or Deligne-Mumford stack requires a lot of definitions and when you think about the structure, you can see why stability is a variable. Then the most difficult thing that appears is, well, imagine that what you need is only this diagram, and you have smooth algebraic manifolds, then you can treat the image of $ev_*(\overline{\mathcal{M}}_{g,n}(V,\beta)) \in A_*(V^n \times \overline{\mathcal{M}}_{g,n})$ and that would be a correspondence. That's the correspondence I'm speaking about for the intuitive meaning. In the category of motives, a motive is a class of algebraic manifold $h(V)$ and a morphism between $h(V) \to h(W)$ is a class $X \in A_*(V \times W)$. If everything would be as smooth as that you would lond in the motivic setting once you know the space $\overline{\mathcal{M}}_{g,n}(V,\beta)$ and this would induce the map of cohomologies.

Now, this is not true, and up to now it is one of the most difficult points of the theory of Gromov-Witten invariants. You should not consider all of $\overline{\mathcal{M}}_{g,n}(V,\beta)$ but rather $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{virt}$, which generally even has different dimension, even in the case where everything is smooth and algebraic. The correspondence is something that embodies the idea of putting things in general position in algebraic geometry. If you vary V, this is a very unstable object, and to put things in general position you should change it. Approximately, imagine, there was a classical problem in a topology, how many zeroes can a vector field on a smooth manifold have? You can ask this in the category of algebraic manifolds. This is extremely unstable and often don't want the precise numer of zeroes. What you should do is consider vector bundles, the tangent bundle, and the last characteristic class of this bundle. This is what you are actually talking about for the zeroes. You produce quite non-trivial theories over spaces related to this one, pass to a characteristic class, and then pass to this one. [pointing]. Please do remember that this kind of diagram, including the virtual fundamental class, is what we have to construct and then study its very interesting properties in order to get to the mid-level version. Even before this, we should produce the two moduli spaces, the absolute and tho one depending on V and β.

There is a lot of freedom for the virtual fundamental class moving instead of to algebraic geometry to symplectic geometry. So up to a certain degree, you can free yourself from these constraints, but then you lose a whole lot of structures on cohomology. Rigidity of algebraic geometry is reflected in a lot of additional structure on the cohomology [long list]. When you connstruct this by algebro-geometric means, you get something much richer. You can try to get these using symplectic geometry instead if you know you can reconstruct from numbers, but I think that whatever refers to the mirror phenomenon is a mirror between algebraic and symplectic, but it's not a very well-stated idea, that way, this is very hazy.

I want now to explain a little bit of what is involved in thinking about moduli spaces and the implicit variable of stability conditions. Several ideas are involved in thinking about it.

First of all, explaining intuitively or drawing the diagram, I was thinking before that point about all the $\overline{\mathcal{M}}$ as "sets of $\mathscr{C}$-points," so I was saying that a point parameterizes a map or a curve. Now, of course, according to Grothiendieck, one should not restrict to points, but to points "with values in variables schemes," often affine schemes but then more complications arise. Then $\overline{\mathcal{M}}(S)$ is $Hom(S, \overline{\mathcal{M}})$. The point is, I have not yet constructed $\overline{\mathcal{M}}$.

As a preliminary step, since I want my $\overline{\mathcal{M}}_{g,n}$, I want the geometric points to be these things, curves with points and so on, let me define $\overline{\mathcal{M}}(S)$ as the set or class of families of objects of a given type "parameterized" by S. So this should be from $\mathscr{C} \to S$, where $\mathscr{C}$ is a scheme and S is our base, so I will want n sections of this, and the question is, should I consider these families or should there be a restriction telling me I want the geometric fibers to depend on S? If the map is only onto a closed subscheme with fibres empty otherwise, then this depenedence is not sufficiently continuous. So I want an idea of continuous dependence on S. This should be sufficiently strong to mean that I don't include things I don't want but sufficiently fluid to allow degeneration. I will get things that are not complete, not projective if I don't allow degenerations.

How do these two conditions combine? The first answer was due to Grothiendieck, and it's quite ungeometric. It's difficult to grasp it intuitively. "Continuity" à la Grothiendieck is the same as "flatness" of π in the sense of algebraic geometry. Suppose you're considering $Spec\ B \to Spec\ A$, a morphism of affine schemes. This corresponds to a map $A \to B$, and then the map $A \to B$ is flat if whenever you have an exact sequence of moduli over A and then you lift it to B, $B \otimes_A M \to B \otimes_A N \to B \otimes_A P$ should remain exact. I was very baffled by this. I didn't see how it corresponds to the idea of a continuous family. One of the best books that enlightened me was the Mumford book, the red book of algebraic geometry or something like that. It explains very carefully why flatness and continuity should be related.

Let's consider this as a tentative definition. Continuous families correspond to flat morphisms, and this should be localized. Now the question is what to do if you want a degeneration. Grothiendieck's idea of how to formalize it was, if you have a family defined over the circle but without the central point. Can you extend a flat family to the central point in a flat way, but usually in infinitely many ways. This is what I will call the $\frac{0}{0}$ phenomenon. The trouble with $\frac{0}{0}$ is not that it has no limit but that it has infinitely many limits. This happens always in the context where you have flat families and nothing more. You can't produce a definition where this exists and is unique. Only if you add a stability condition will this be tamed. It won't be unique, but you can uniquely extend over a hole. That was the point of much previous work. Stability conditions were invented and studied to classify vector bundles. These were produced until, some time after 2000, it was stressed that one should consider different stability conditions which form an interesting space which has walls so that if you see it in a chamber, you do nothing different but when you cross a wall the compactification changes. Let me just show two examples of stability conditions directly relevant for our purposes.

Stable curves with n labeled points of genus 0:

Stability condition 1 (this is the most common). Consider a curve which is either $\mathbb{P}^1$ with at least three marked points that are not equal or a tree of such curves where the labeled points are never singular. One can show that this is equivalent to saying the automorphism group of such a tree is actually identical. There is only the identity isomorphism.

Stability condition 2 First, $x_1 \neq x_n$ and nonsingular. The remaining points can be pairwise coincide but not coincide with x_1 or x_n. If it's not $\mathbb{P}^1$, it's not a tree but a sequence and x_1 and x_n are at the ends. On each component there should be at least three singular points. This again ensures that the automorphism group is just 1.

One can prove that there is an $\overline{\mathcal{M}}_{0,n}$ for each of these, they are very different. This second one $\overline{L}_{0,n}$ is a well-known toric manifold. In this case, one can prove that the Gromov-Witten invariants are combinations of the Gromov-Witten invariants for the usual story.

Flatness allow you to have continuity, and stability gives you a definite meaning of expressions of type $\frac{0}{0}$.

[By Costello, if Hodge to de Rham degenerates, you can get GW invariants from a Calabi-Yau category]. Toen answered my question about motives generated by [unintelligible]stacks. It turned out that the functors sending your manifold or stack to its motive is large. Therefore you can extend any standard cohomology theory to the cohomology theory of stacks in a nonobvious way. I think putting together a lot of papers there is a complete theory of GW invariants in DM stacks. I don't think there is a paper where everything is stated clearly. For mixed motives I don't know.