YURI MANIN

GABRIEL C. DRUMMOND-COLE

Okay, so let me recall for you that basically I interpreted last time the nonlinear differential equations involved in the F-identity and the metrics as integrability conditions. I'll get a system of equations that has as many solutions as it can. Then it's helpful to say that the solutions can be given by initial conditions. So in the semisimple case, if you have a germ (M, x_0) of a Frobenius manifold, then it is uniquely defined by the following finite amount of data:

First of all we want canonical coordinates $u_i(x_0)$ at this point x_0. Saying it like this I'm saying basically nothing because these are defined up to multiplication and addition, so a point has any canonical coordinates you wish. This becomes unique only if you have given the Euler field E. Then the canonical coordinates are eigenvalues of $E\circ$ at $T_{x_0}M$. The eigenvectors are e_{i,x_0} so that $e_{i,x_0}e_{j,x_0} = \delta_{ij}e_{i,x_0}$. I need the coefficients of the compatible flat metric $\eta_i(x)$. We know that there is a function metric potential η so that $e_i\eta = \eta_i$. I know this only at one point, and at the same point the second derivatives $\eta_{ij}(x_0)$, and this is sufficient to define the whole germ, but you see a certain stupid choice is involved, if the i are 1 to n then a numbering is involved of the u_i. Speaking only about a germ you can pay no attention at all, but for a global everywhere semisimple Frobenius manifold it might have nontrivial fundamental group, so you have monodromy, you'd better assume that this is simply connected and then the number doesn't matter again.

There are some more or less trivial algebraic restrictions. A semisimple germ is defined by this finite data. I will use this fact to show you the mirror picture for projective spaces $\mathbb{P}^r$. I will look separately at the quantum cohomology for projective spaces, and I will find points where it is semisimple, and then I will construct a Saito structure, I will choose a point, calculate the germ and show that they coincide. That will give us the mirror picture. Actually, I will give almost complete constructions of u_i and η_i and omit the η_{ij}, discussing it a little.

Let us start with $\mathbb{P}^r$. As I already explained several times, we have the flat coordinates x_a where a will run from 0 to r and x_a will be dual to the cohomology basis Δ_a which is dual to $[\mathbb{P}^{r-a}]$ in $H^*(\mathbb{P}^r)$. You can put things geometrically as $\Delta_a = \frac{\partial}{\partial x_a}$, and then the main information from Gromov-Witten invariants is encoded in the potential

$$\Phi(x) = \frac{1}{6}\sum_{a_1+a_2+a_3=r} x_{a_1}x_{a_2}x_{a_3} + \sum_{d=1}^{\infty}\left(\sum_{n=2}^{\infty}\sum_{a_1+\cdots+a_n=r(d+1)+d-3+n),a_i\geq 2} I(d,a_2,\ldots,a_n)\frac{x_{a_1}\cdots x_{a_n}}{n!}\right)e^{dx_1}$$

I start counting when $a_i \geq 2$ because $a_i = 0$ is only classical and $a_i = 1$ is in e^{dx_1}. The $I(d, a_1, \ldots, a_n) = \langle I^{\mathbb{P}^r}_{0,n,\beta=d\Delta_{r-1}}\rangle(\Delta_{a_1}\otimes\cdots\Delta_{a_n})$.

Then we get a new multiplication $\partial_a \circ \partial_b = \sum \Phi^c_{ab}\partial_c$. So the first part is the classical part and then the second part is the quantum correction. Only a common point contributes to the intersection, but in the quantum case if there is a rational curve between them this also contributes to the intersection, but this gets smaller and smaller with the growth of the degree of this curve.

So we know $\circ$ multiplication in flat coordinates. One easy corollary of these formulas is that along H^2, that is where $x_0 = x_2 = x_3 = \cdots = 0$, the multiplication is semisimple. You will see by induction everything is generated by ∂_1, and the last power which was 0 in the classical case is not zero $\partial_1^{(r+1)} = e^{x_1}$. So semisimplicity is there, and I will be able to calculate things exactly along H^2. There is a general term. I am putting myself on H^2 only, the resulting ring is called small quantum cohomology. I am multiplying tangent vectors which might not be in H^2 but projecting the result to H^2. The quantum cohomology is semisimple there. We can define the very standard idempotents. So $\mathbb{C}[x]/f(x)$, the idempotents correspond to normalized linear factors.

You should imagine that, this manifold has its own coordinates. Imagine that the series I wrote converges around 0. On this germ I have flat coordinates x_a. I can take the tangent space at any point, and calculate structure constants at that tangent space, and if $z = \partial_1$ then you will get $\mathbb{C}[z]/(z^{r+1} - 1)$ when $\partial_1 = 0$ and everything else is 0.

The semisimple algebras are not deformable, so in the semisimple part you will only get something new if you look at the basis of semisimplicity. You have two kind of specific basis, the flat coordinates and the canonical coordinates. In canonical coordinates the multiplication is the same; in flat coordinates it changes.

In an abstract commutative semisimple algebra over $\mathbb{C}$ you have idempotents e_i which are canonical up to choosing order. If you choose a different basis, then multiplication will be different. The dependence is exactly dependence of the third derivative. If you want, you can say that ∂_a is given in terms of e_i. This implicit formula for transformation of flat coordinates to canonical ones can develop singularities, and so with no canonical coordinates you lose semisimplicity and with no flat coordinates your [unintelligible]cannot be analytically continued.

The formulas to get canonical coordinates from this side, the point $(x_0, x_1, 0, \cdots)$, any point of this form in flat coordinates has canonical coordinates $u_i = x_0 + \zeta^i(r+1)e^{\frac{x_1}{r+1}}$ where ζ is a root of unity of degree $r+1$. These are canonical coordinates along H^2. In the case of quantum cohomology, it is easy to calculate the metric potential η, and $\eta = x_r$ and therefore we can basically derivate this and get that $\eta_i = \frac{\zeta^i}{r+1}e^{-x_1\frac{r}{r+1}}$, this is the flat metric in canonical coordinates. For the second derivatives in two different directions $k \neq i$, $\eta_{ki} = -2\frac{\eta^{i-k}}{(\eta^{i-k}-1)^2}\frac{e^{-x_1}}{(r+1)^2}$. To calculate the first data it's enough to set all the coordinates past the first two equal to zero. Then to go on to the second derivative you need to know the linear parts of the quantum cohomology. So initially you should take into account monomials of degree 4, so Gromov-Witten invariants with 4 arguments. But you can calculate this since one line goes through 2 points. Then all of them are 1 and you can make calculations involving some algebraic identities and this is what you get as your answer.

Now let's look at the mirror dual picture. I will have to take a function and deform it. There's a nice theory when there is an isolated singularity at $x = 0$. For quantum cohomology it simply does not work. We have to take the function $F(z, 0)$ (later I will put parameters of deformation) to be $z_1 + \cdots + z_r + \frac{1}{z_1 \cdots z_r}$. A similar way works for weighted projective spaces, other kinds of spaces, then you should produce a deformation, and the general trick is to treat this as if there was an isolated singularity. Construct the ring $\mathbb{C}[z_i^{\pm 1}]/(\frac{\partial F(z,0)}{\partial z_i})$ and take the linear combination with arbitrary coefficients.

I will not do this here, it's not the most convenient example to use. I'll pick a particular deformation, with one parameter q, so I'll take $F(z, q, 0, \ldots 0) = z_1 + \cdots + z_r + \frac{q}{z_1 \cdots z_r}$. It will turn out that the quantum multiplication point where $q = 1$ and everything else is 0 or q is generic is semisimple, it's just multiplication in the Milnor ring. Again I will be able to do the calculation for first derivatives. The formulas for the initial conditions will have $q = e^{x_1}$, so variation along q will correspond to variation on H^2. In more complicated examples the deformations are not trivial.

Let me see what I should calculate. What can I do right away? I first must calculate the ideal of derivatives. The first derivative is

$$\frac{\partial F}{\partial z_i} = 1 - \frac{q}{(z_1 \cdots, z_r) z_i}.$$

At critical points, all the products are equal to q, so $z^{r+1} = q$. There are $r + 1$ possible values which differ from one another by a root of unity. We get $(\zeta^i q^{\frac{1}{r+1}}, \ldots, \zeta^i q^{\frac{1}{r+1}})$. Then the canonical coordinates are the values of F at these critical points $F(\rho_i)$, and I am adding these things, I get $r + 1$ copies of this, and then divide by the product, it's all easily calculated, and I get $F(\rho_i) = \zeta^i (r + 1) q^{\frac{1}{r+1}}$. If you compare that with the quantum cohomology. If I identify q and e^{x_1}, I get this correspondence.

Then the coefficients of the flat metric η_i, I had written the flat metric for a Saito structure after the choice of Saito's primitive form, and I should say here the primitive form is, and this seems to be the general rule, it is the differential of the first kind on the torus, where all the z_i are nonzero, $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_r}{z_r}$, and I can easily prove that the η_i are $\frac{1}{\det(z_i z_j \frac{\partial^2 F}{\partial z_i \partial z_j}(\rho_i))}$ [as long as you have some aside about primitive forms and logarithms that I missed.]

This is $\frac{\zeta^a}{r+1} q^{-\frac{r}{r+1}}$. If $i = j$ then you have z_i^2, you derivate it again and get a 2, and after all you will have to calculate the determinant of the matrix with 2 on the diagonal and 1 everywhere else. As I did here I will omit the story and I'll get the same result with η_{ij}, if you input q instead of e^{x_1}. The step requires taking into account polynomials of degree 4 in quantum cohomology or additional parameters of deformation in the Saito structure.

This is the mirror example in the generically semisimple case. The generic semisimple case has as necessary Hodge of type p, p, and there is no known sufficient conditions.

[missed some]

I will briefly explain what I will be doing in the remaining part of the course, which is explaining the general idea of the big algebraic-geometric panorama of quantum cohomology. What is difficult about this story is that one needs to seriously enlarge the scope of algebraic geometry, but consider Deligne-Mumford stacks. I'll try to explain to you the intuition behind the things. It's not so hard to define but then we need the theory of algebraic cycles and intersection theory and various new techniques. I'll try to explain why they are really necessary for Gromov-Witten invariants and what will come at the end. There will be no proofs practically but explanations about what we will learn about all algebro-geometric manifolds that we did not know before.