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I mentioned that quantum cohomology generally refers to understanding maps $I_\beta^V : H^(V)^{\otimes n} \to H^*(\overline{\mathscr{M}}_{n,g})$. There are infinitely many of these, but you have a bunch of variables. You have β in a positive cone in $H_2(V,\mathbb{Z})$, you have variables n and g. You have an implicit variable of the stability condition, which traditionally is fixed once and for all. There are various ways of squeezing this information, restricting, cutting down. One traditional way to cut down is to restrict to the case $g = 0$ and another probably to the case where, only numerical invariants, $\langle I_{g,n}^V \rangle$ integrating over the fundamental class. We should consider the virtual fundamental class as created, although I did not do so.

Now that this information is cut down, we can encode it into an analytic or differential geometric object. I had in mind F or Frobenius manifolds, and I will explain if I have time very briefly a way to cut it down even more, the so-called J-function of Givental, constructed in a different way. Then you should ask, "what did I lose?" There are two basic questions. First of all, what you can do if you take only $g = 0$ information. Another is if you consider only numerical information, can you reconstruct the whole motivic story? These are reconstruction problems. Then I mentioned the word motives, implying or reminding that H^* is also variable, you can work in any cohomology theory. Therefore you should be able to work in universal cohomology theory, so in motives? In fact, the fact that one can put instead of H, the letter h that means motivic cohomology because this is an algebraic correspondence. So then one can look at this categorically.

You have a lot of invariants, you can look at parts of it, you can look at encodings, these analytic pieces are some kind of Feynman integral, and this encoding is a small part about how to make this precise. What physicists really mean is not well-understood.

Now let me illustrate some parts of this general picture, for example, when $V = \overline{\mathscr{M}}_{0,m}$. The structure is pretty well-understood. As I said, the calculation of quantum cohomology is very much an unsolved problem, even restricting to genus zero. There are a lot of subproblems. So we have $I_\beta^{\overline{\mathscr{M}}} : (\overline{\mathscr{M}}_{0,m})^{\otimes n} \to H^*(\overline{\mathscr{M}}_{0,n})$. The first step is to understand the players here. Already with β there is a lot of trouble. So $\beta \in H_2(\overline{\mathscr{M}}_{0,m},\mathbb{Z})$ and also in the positive cone. The positive cone is the first unknown in this story. Let me give you more concrete information.

What is $\overline{\mathscr{M}}_{0,n}$? There exist curves of genus 0 only for $n \geq 3$, then the dimension is $n-3$, since the top cell is something like $(\mathbb{P}^1)^n$ with deleted diagonals modulo $Aut\ \mathbb{P}^1$. This can be made more precise. We know too that $H^2(\overline{\mathscr{M}}_{0,n},\mathbb{Z}) = A^1(\overline{\mathscr{M}}_{0,n},\mathbb{Z}) \cong \mathbb{Z}[\underbrace{\quad D_\sigma \quad}_{\text{boundary divisors}}]/(R)$ The relations correspond to splitting into two components and distributing along those two components. So σ corresponds to splitting n into two components, S_1 and S_2 which each have at least 2 points. Further degeneration is then allowed. There are as many divisors here as partitions of this kind. The relations were established by Keel. There are two kinds of them. Say you have σ and

τ two partitions, then we can consider, they are unordered, S_1, S_2 is the same as S_2, S_1. We define $a(\sigma,\tau)$ is the number of nonempty pairwise intersections between S_i and T_j. There are no more than four and at least two, so it's 2, 3, or 4. If it is 2, then $\sigma = \tau$ Otherwise it's three or four. There are i, j, k, ℓ so that $ij\sigma k\ell$ and $ik\tau j\ell$. Then if $a(\sigma,\tau) = 4$, then the product $D_\sigma D_\tau$ belongs to the ideal (R). For any i, j, k, ℓ, we have $R_{ijk\ell} = \sum_{ij\sigma k\ell} D_\sigma - \sum_{ik\tau j\ell} D_\tau$ is in the ideal. Keel showed that these applied and Maxim and I showed that this was it. So this is all generated by the boundaries. This is a combinatorial thing that is known, so you should pass to the left-hand side $H^*(\overline{\mathscr{M}}_{0,m})^{\otimes n}$ and see what happens to the generators. It's good to start with small values of m and n. I didn't see any calculation of this sort in the literature, but it's a good introductory exercise. It's difficult to hope that we will somehow know a nice general formula, and in particular, there is a problem of the positive cone, we really don't know what it is. It's a particular case of a well-known phenomenon. These sholud have nonnegative intersections with divisors. We know boundary generators of $H_2(\overline{\mathscr{M}}_{0,m}, \mathbb{Z}) = A_1(\overline{\mathscr{M}}_{0,m})$, which are generated by the trees of degenerated curves so that there are exactly three points on each curve. You have these degenerate rigid stable curves, these are the points, I'm sorry, and the cycles corresponding to curves at the boundary will allow just one component where there are four of these. We know generators, and we know generators D_σ and we want to come up with linear combinations of the first generators which have nonnegative intersection index with D_σ. Probably there are other effective divisors. It's known according to Mori theory that the possible behavior of the cone depends on whether minus canonical class is [unintelligible]or not. The anticanonical class, starting with six or seven, becomes non-effective. In principle, Mori theory allows a nonpolyhedral part of the boundary. There are various infinity possibilities. It's important to know what is this cone β.

I know of two important papers devoted to this story? Castravet, Exceptional loci on $\overline{\mathscr{M}}_{0,n}$ and hypergraph curves. I don't have the other one here.

So what about the encoding. You have $H^*_{quant}(\overline{\mathscr{M}}_{0,n})$, what is that? It involves only the genus zero picture, and you know several questions that are unsolved. You know now that H^2 generates the whole quantum cohomology, so one can use the first reconstruction theorem, which basically tells you it suffices to consider the triple correlators $\langle * * * \rangle^V_\beta$, but you don't know much about β. But in principle this gives you everything. Then you can ask whether you can apply the other reconstruction theorem, whether the quantum cohomology is semisimple. Nobody knows. If it is, you could try to calculate at a certain point, special coordinates, so on. Then you may try to ask for Givental's generating function, and I will explain a little bit of this. First of all, generalities.

Basically, Givental considers the case, take Y a smooth projective manifold, and we will say smooth Fano. The object that will be considered is not necessarily Fano. So we consider when ω_Y^{-1} is ample (which is not true in our case but it won't matter) and the J-function is a series of the following kind: it is a series in one variable t, and it is a double sum

$$J(t) = \sum_{d\geq 0} \sum_{\gamma_i} \sum_{\beta:(-K_Y,\beta)=d} \langle I^Y_{0,3}\rangle(\gamma_1\otimes\gamma_2\otimes\gamma_3)t^d.$$

In quantum cohomology we have q^β, here we squeeze this down to taking into account only $(-K_V, \beta)$. Then as a coefficient, you sum all Gromov-Witten invariants that appear. Instead of information about these numbers separately, you have certain sums. What happens is, it is very convenient for encoding this Gromov-Witten information analytically and providing a "weak mirror" picture, for example, for toric Fano varieties.

Can this be extended to when this is no longer ample, so to $\overline{\mathscr{M}}_{0,n}$? Let me show you the simplest mirror phenomenon that can be demonstrated in this way. The mirror phenomenon I was considering before that, if you have a quantum cohomology H^*_{quant} and a certain Saito unfolding picture on the other hand, then I have demonstrated [unintelligible], a class of unfoldings of Laurent polynomials. This is a mirror of the first type. It turns out that if one considers in the same situation the J function instead of H^*_{quant} and on the other side something related to such a Laurent polynomial, taken at a certain point, then this mirror can be described as a quality. We don't have the unfolding and versal unfolding and so on, we don't have that but what he have is a simple thing. Consider a torus. A Laurent polynomial is a function on a torus, which is the spectrum of the ring of the Laurent polynomials. Consider the universal differential form $\omega = \wedge \frac{dx_k}{2\pi i x_k}$. So if you have a w, you can integrate

$$\int_{|x_i|=\epsilon} \frac{\omega}{1-tw} = \int_{|x_i|=\epsilon} \left(\sum_{i=0}^{\infty} tw^i\right) \omega.$$

This is a power series in t has as coefficients the constant term of w^i. This is the J function. This is a weak mirror phenomenon. The question is when that corresponds to the Saito mirror phenomenon. This is a popular version of the mirror, because Givental made a lot of very clever contributions. He has never looked at Frobenius manifolds, so people on one side don't look at the other side.

Again returning to $H^*(\overline{\mathscr{M}}_{0,n})$, we don't know if either formalism can be meaningfully applied, because it's not Fano, The quantum cohomology must be very basic. We don't know what it is.

Finally, I want to make a few comments about a very different question related to mirrors in which it is essential to understand them at the motivic level. Here something very unusual happens that was actually observed in the initial stages but never well-understood. If you're considering three-dimensional Calabi-Yau manifolds, and consider their cohomology, you have the Hodge-Tate type and then you have the $p+q=3$ part on the other side. If you go to the mirror side, the horizontal becomes the vertical and vice versa. This cannot be so if the mirror map itself is motivic. How can it be, then? Even to ask this question in a nice way is not very easy, and it seems to me now that some kind of unifying assumption is this: on the left hand side, try to consider manifolds with only Hodge-Tate cohomology, or whose full quantum cohomology is semisimple. In particular, $\overline{\mathscr{M}}_{0,n}$ satisfies this. Then their mirrors should be arithmetic but [unintelligible]. [unintelligible]tried this on the level where you have a Fano variety and you look only at the vertical part, and it turns out that in the Givental language you get hypergeometric functions. There is a lot of interesting classification. This would be fantastically interesting for $\overline{\mathscr{M}}_{0,n}$. I'm finishing now. My main goal today was to show you what kind of boundary of the known territory can be considered as a source for new problems and ways of thinking. Any questions? This was my last class at Northwestern. I wanted to thank you, I enjoyed it very much, and farewell.