YURI MANIN

GABRIEL C. DRUMMOND-COLE

Today I will explain some more details about structure needed for F-manifolds and Frobenius manifolds. The subject will be the structure connection. Let me remind you that we are considering manifolds M with an $\mathscr{O}_M$-bilinear product $\circ$ on the tangent sheaf, and sometimes with a metric which should be flat, ∇_0, and there is compatibility and the F-identity. The aim of this lecture will be to encode all of this in the properties of some connection, not just a connection but a one-dimensional family of connections. Before doing this, I will remind you some preliminaries of the general properties of connections. These will be true for any reasonable category of manifolds. I'll think of analytic varieties over $\mathbb{C}$, but much of this will work wherever. If you have a vector bundle E over M (think of it as a locally free sheaf) then a connection ∇ is an operator $E \to \Omega^1_M \otimes_{\mathscr{O}_M} E$. This should satisfy the Leibniz identity $\nabla(fs) = df \otimes s + f\nabla s$. Instead you can do covariant derivatives along vector fields ∇_X which will each send $E \to E$, and the Leibniz identity will look like $\nabla_X(fs) = Xf \cdots s + f\nabla_X S$.

If you have two connections ∇_1 and ∇_2 then $\nabla_1 - \nabla_2 : E \to E$ will be $\mathscr{O}_M$-linear, and vice versa, if $A \in \Omega^1_M \otimes End_{\mathscr{O}_M} E$ and given a connection ∇_1, I can define $\nabla_2 = \nabla_1 + A$, and this will be a connection. It's easy to check that I will have the Leibniz identity. Then the space of connections on E form a principal homogeneous space.

Now what is very important is that given any ∇, it uniquely extends to $\tilde{\nabla} : \Omega^\bullet_M \otimes E \to \Omega^{\bullet+1}_M \otimes E$ where the Leibniz formula looks like $\nabla(\omega \otimes s) = d\omega \otimes s + \omega \otimes \tilde{\nabla} s$, and then if I take $\tilde{\nabla}^2$, it will be already a purely linear operator, $\tilde{\nabla}^2$ is $\mathscr{O}_M$-linear , and just multiplication by what? It should be a form of degree two. It should be applicable to any section so there is a component coming from $E \to E$. It will be a form R in $\Omega^2_M \otimes End_E$. This is called [unintelligible]curvature. The most important part of it is that there is an equivalence between two statements, curvature is 0 if and only if the kernel of ∇ is the sheaf of linear spaces, $E^f \subset E$. Now it's necessary to imagine the language of differential equations, $\nabla S = 0$ is a differential equation of a first order. Any vector in the fiber can be a solution, and there is a unique flat fiber (?) continuing this solution. If $R \neq 0$ there might be no solution. The unique correspondence between such solutions and initial conditions corresponds to the case when the curvature is zero.

I've only been considering connections on an external vector bundle. It's a good exercise to write everything locally in terms of a chosen basis of E and local coordinates on M. You'll see that I'm considering differential equations of the first order. Then everything becomes partially algebra and partially easy first order differential equations.

In this case my E is actually $\mathscr{T}$. I'm applying all of this to $E = \mathscr{T}_M$, and some additional operations and possibilities for interactions appear. What will be for me essential, there is another tensor, called "torsion." In terms of vector fields it sends a pair $(X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$. Also it is antisymmetric bilinear. This tensor then belongs to Ω^2_M, and one can assume that something interesting happens when it is zero, and it is in fact equivalent to the fact that ker ∇, (assuming that $R_\nabla = 0$), the bracket is zero there. Then $\mathscr{T}_M^f$ is a sheaf of Abelian Lie algebras.

Now something surprising We can encode all of this [F-manifold stuff] into vanishing of curvature and torsion. I will explain this encoding. Suppose that I have $(M, (\mathscr{T}_M, \circ), \nabla_0)$, assuming nothing on $\circ$, I can produce a pencile of connections on $\mathscr{T}_M$ of the form

$$\nabla_\lambda(Y) = \nabla_{0,X}(Y) + \lambda X \circ Y,$$

a one-parameter family of connections, and conversely, the reverse is true. Given a pencil of connections $\nabla_0 + \lambda_A$, $A \in \Omega^1_M \otimes_{\mathscr{O}_M} End_{\mathscr{O}_M} \mathscr{T}_M$, and multiplication $X \circ^A Y = i_X(A)(Y)$ where $i_X(df) = Xf$.

I did not take into account the flatness or F-identities or anything. The marvelous theorem is that

Theorem 1. (1) *$\circ^A$ is commutative if ∇_λ has vanishing torsion for all λ.*

(2) *Assuming that this assumption holds, and that $\nabla_0^2 = 0$, then when I restrict $\circ^A$ to ∇_0-flat vector fields, it can be written as $X \circ^A Y = [X, [Y, C]]$ if and only if in the equation $\nabla_\lambda^2 = \lambda R_1 + \lambda^2 R_2$ we have $R_1 = 0$.*

(3) *Then we have associativity, $\circ^A$ is associative if and only if $R_2 = 0$*

Generally when you have nonlinear differential equations it's hard to come up with the appropriate conditions. But some of these are integrability conditions. The precise and typical case is to establish that the non-linear differential equations are equivalent to the fact that some related linear differential equations have as many solutions as they can. The nonlinear ones will be like a moduli space for the non-linear ones. These had big applications in the 60s and 70s. In fact, this whole story about deep-water waves leads to infinite dimensional F-manifolds. There are papers in the ArXiv showing connections from that story to this one.

As soon as you know the statement the check is pretty formal. For example, about the first claim, you write

$$\nabla_{\lambda,X} Y = \nabla_{0,X}(Y) + \lambda X \circ^A Y$$

and taking a basis (∂_a) of local flat vector fields with respect to ∇_0, I get that the torsion of ∇ vanishes means that $\nabla_{\lambda,\partial_a}\partial_b = \nabla_{\lambda,\partial b}\partial_a$ if and only if $\partial_a \circ \partial_b = \partial_b \circ \partial_a$.

The condition $[X, [Y, C]]$ implies the F-identity along with compatibility. So write $\partial_a \circ \partial_b = \sum A^c_{ab}\partial_c$. We know that this is commutative, so $A^c_{ab} = A^c_{ba}$. Now $R_1 = 0$. means that $\partial_a A^e_{bc} = \partial_b A^e_{ac}$, which is the same as the fact $\sum_b dx^b A^e_{bc}$ is closed, and it is thus a complee differential. Then there is locally a B^e_c so that $A^e_{bc} = \partial_b B^e_c$, and then from the symmetry of A^e_{bc} in b and c, it turns out that $\sum dx^c B^e_c$ is closed, so it's also a differential, so that we can get C^e so that $B^e_c = \partial_c C^e$ Then it turns out that this presents C as $\sum C^e \partial_e$.

One more miracle happens. We've introduced extra structure of Euler fields. These can also be encoded in the connection, but we should consider a slightly larger manifold. Assume we have M and $(\circ, e)$. Pick local coordinates (x_a) on M, and produce the manifold $\hat{M} = M \times (\lambda - \text{line})$, where this last is interpreted as the situation demands, and consider the tangent sheaf on the enlarged M, considering vertical vector fields perhaps depending on λ. Then I have covariant derivatives in the previous picture, so I have all the previous conditions satisfied, I have an F manifold, and I have covariant derviatives $\hat{\nabla}_{\lambda,X}$ which so far do not depend on λ. So $\hat{\nabla}_{\lambda,X}(Y) = \nabla_{0,X} Y + \lambda X \circ Y$. What I would like to add is the covariant derivative with respect to this coordinate $\hat{\nabla}_{\frac{\partial}{\partial\lambda}}(Y) = \frac{\partial Y}{\partial\lambda} + E \circ Y + \frac{1}{\lambda}(\nabla_{0,Y}E - Y)$ for a fixed vector field E. Now the theorem says that $\hat{\nabla}$ with E independent of λ is flat if and only if E is an Euler field of weight 1 on the F-manifold M. In terms of curvature and torsion of connections, in a sense, a vector bundle consisting of vertical vector fields, so everything is translated in terms of flatness and torsionlessness of certain vector fields. I want to formulate a small problem that I several times tried to do somethinga about. I could not summon the degree of creativity needed to solve this. We know that $\circ : \mathscr{T}_M \otimes \mathscr{T}_M \to \mathscr{T}_M$ is one of our main players. We could instead consider $\Omega^1_M \to \Omega^1_M \otimes \Omega^1_M$. Can we translate the F-identity in a meaningful way? What is the meaning of the F-identity?

Now I will briefly explain the way that by explaining the integrability of linear equations one can do something with the nonlinear ones, and apply this to semisimple Frobenius manifolds. In this case the answer is easy.

A short application to semisimple Frobenius manifolds. We know that semisimple F-manifolds are very easy locally. Everywhere you have a system of coordinates u_a so that $e_a = \frac{\partial}{\partial u_a}$, $e_a \circ e_b = \delta_{ab} e_a$ and $e = \sum_a e_a$ with $E = \sum u_a e_a$. Suppose you want to add a flat metric. Then, of course, you must have a quasiRiemannian metric, and for it to be invariant with respect to multiplication, you easily see that $g(e_i, e_k) = g(e_i \circ e_i, e_k) = g(e_i, e_i \circ e_k)$ which is 0 unless $i = k$. So the metric is diagonal in these idempotents. So $g(e_i, e_i) = \eta_i$. I took into account the multiplication and Frobenius property but not the flatness.

Theorem 2. *Flatness of such a g is equivalent to the following system of equations*

(1) $\eta_i = e_i \eta$ *for a single function η So $\sum \eta_i du_i$ is closed, it is $d\eta$.*
(2) *Let $\eta_{ij} = \frac{\partial^2}{\partial u_i \partial u_j}$ and write $\gamma_{ij} = \frac{1}{2}\frac{\eta_{ij}}{\sqrt{\eta_i}\sqrt{\eta_j}}$ (where I must make choices locally of branches of the square root). Then the Darboux-Egoroff equations must be satisfied: $e_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}$ and $(\sum e_a)\gamma_{ij} = 0$ where no two of the i, j, k are equal. We have one unknown function η, the equations are of the third order, and the right-hand side is quadratic in the second derivative, mildly nonlinear. What kind of initial data should define the analytic continuation? The answer is that you should give the first derivatives, second derivatives, and the point itself should be given by values of u_i. You may need compatibilities, so intuitively you would expect that given this kind of initial data with obvious compatibility conditions you would be able to produce a function η that would solve this differential equation and produce the Frobenius structure on the given F-manifold. This is so, up to several funny and non-obvious restrictions. The intuition is that the u_i are the coordinates in which the multiplication is simple. The η is the coordinates in which the metric is simple. The transition functions can develop singularities wherever the canonical coordinates, well, you could develop a pole. Flat coordinates seems to be so innocent. u can develop singularities. This is not well-understood or well-studied. If you do not go to non-semisimple points, you're okay, but as soon as you go to the boundary and lose your canonical coordinates, it's unclear in which terms your things should be described. I will suggest that a good way to approach this question is to apply stability conditions. The only thing I want to say, the last sentence, when you're proving that these numbers suffice, the best way to see this is to interpret integrability conditions by way of connections.*