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GRADING EVENT, PART II

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1. Reconstruction

For my part, I will only consider the genus zero situation, so they are maps, if V is a smooth projective variety over $\mathbb{C}$, maps $I^V_{0,n,\beta} : H^*(V, \mathbb{Q})^{\otimes n} \to H^*(\mathscr{M}_{0,n}, \mathbb{Q})$ for $n \geq 3$ and $\beta \in H_2(V, \mathbb{Z})$. My topic is the first reconstruction theorem. It's very easy to state, so let me state it right away.

Theorem 1. *If $H^*(V)$ is generated by $H^2(V)$, then any system $\{I^V_{0,n,\beta}\}$ is uniquely determined by $I^V_{0,3,\beta}$, and I can further assume that one of the arguments is in $H^2(V)$.*

I want to talk about some of the things we need to be able to prove this theorem. One thing to understand is what $\overline{\mathscr{M}}_{0,n}$ is.

The variety $\overline{\mathscr{M}}_{0,n}$. This is supposed to be the moduli space of rational curves with n marked points. How do you define this? Let k be algebraically closed. A stable n-pointed rational curve over k is a curve with n distinct smooth points $x_1, \ldots, x_n$ satisfying the conditions that

(1) the genus is 0,
(2) the singular points are ordinary double points,
(3) each irreducible component is a copy of $\mathbb{P}^1$,
(4) on each irreducible component, the number of marked points and singular points together is at lesat three

It's very easy to see what such a curve looks like. If it's smooth it must be P^1. In general, it's a tree of $\mathbb{P}^1$, you can't have any cycles. Then there are marked points. [picture]

The point of the conditions is that such a curve cannot have any automorphisms. It has to preserve the marked points and then you can see it preserves the singular points as well. The only morphism fixing three points of $\mathbb{P}^1$ is the identity.

We have this notion for a curve over k, and now if you want to generalize this to families, let S be a scheme over $\mathbb{C}$. We define $\overline{\mathscr{M}}_{0,n}(S)$ to be a set of maps $C \to S$ with n sections that is flat and proper, and so that each geometric fiber is a stable and pointed rational curve. I look at isomorphism classes of these objects.

Theorem 2. *$\overline{\mathscr{M}}_{0,n}$ is a smooth projective variety over $\mathbb{C}$.*

This was proved by Knudsen. That means that it is, it's a contravariant functor of S represented by a smooth variety. I should say this, for $n \geq 3$. For example $\overline{\mathscr{M}}_{0,3}$, this is just a point. I have only three marked points. I can't have more than one irreducible component in my curve.

$\overline{\mathscr{M}}_{0,4}$ now, I have this $\mathbb{P}^1$ with three marked points, I have a fourth smooth point. It can be any point on $\mathbb{P}^1$ except those three, this means that if I remove the bar, $\mathscr{M}_{0,4}$ is nonsingular curves, this is $\mathbb{P}^1$ minus three points. This is Zariski open in $\overline{\mathscr{M}}_{0,4}$. If I arrive at one of the three points it creates a singular branch. There is a unique isomorphism between any two of those curves. I'll add a singular point. So $\overline{\mathscr{M}}_{0,4}$ is $\mathbb{P}^1$ over $\mathbb{C}$. If you look at $\overline{\mathscr{M}}_{0,5}$ you will always have $\mathscr{M}_{0,5}$ inside of it, so generally $\mathscr{M}_{0,n} = (\mathbb{P}^1 - \{a,b,c\})^{n-3}$ with the diagonals deleted, so in particular it is dimension $n-3$. If S_1, S_2 is a partition of $\{1,\dots,n\}$ with $|S_i| = n_i \geq 2$, then $\overline{\mathscr{M}}_{0,n_1+1} \times \overline{\mathscr{M}}_{0,n_2+1} \to \overline{\mathscr{M}}_{0,n}$ that identifies the extra marked points.

This is a closed immersion. Let me call it φ_S for the partition S. In fact, the images of φ_S are the irreducible components of $\delta\overline{\mathscr{M}}_{0,n}$. This is the moduli space of singular curves. In the case $n = 4$ I have three partitions of $\{1,\dots 4\}$ corresponding to these three points.

This gives me a family of prime divisors.

Theorem 3. (1) *The Chow ring of this variety is isomorphic to its cohomology ring,* $A^*(\mathscr{M}_{0,n}) \cong H^*(\mathscr{M}_{0,n})$.
(2) *In fact, the ring is generated by the image of map* φ_S.

I'll just maybe give, I won't state the axioms for Gromov-Witten invariants, but I'll state one of them, the splitting axiom, which says

$$\varphi_S^* I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \pm \sum_{\beta_1+\beta_2=\beta} (I^V_{0,n_1+1,\beta_1} \otimes I^V_{0,n_2+1,\beta_2})(\bigotimes_{S_1} \gamma_j \otimes \Delta \otimes \bigotimes_{S_2} \gamma_j)$$

This is an incarnation of saying that you can count degenerations.

Theorem 4. *An auxilliary result, not weaker, no hypothesis on V, if V is arbitrary, then $\{I^V_{0,n,\beta}\}$ is determined by its codimension 0 classes, those classes that end up in the top cohomology group of $\overline{\mathscr{M}}_{0,n}$, the numbers that we consider, only the top cohomology part.*

Maybe I can quickly prove this using the splitting axiom. I believe it is not true at all in higher genus. The cohomology ring is generated by φ_S, and the splitting axiom tells me about what happens on divisors. By this axiom, we can express each class in terms of smaller order classes. Then I use the fact that the φ_S generate the whole ring. Maybe that's enough.

2. Reconstruction Theorem II

First let me write down the statement of the theorem.

Theorem 5. *Assume that the (p,p)-part of $H^*(V)$ is generically semisimple and admits a tame semisimple point in $H^2(V)$. Call this point o. In this case all genus 0 Gromov-Witten invariants can be reconstructed from $\langle\gamma_1,\dots\gamma_n\rangle_\beta, n \leq 4$, where $\langle\gamma_1,\dots,\gamma_n\rangle$ means $I^V_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$.*

[The first part is equivalent to there being no nontrivial part when $p \neq q$.]

From the dimension axiom we get that $k(\beta) = (-K_v, \beta) = \sum(|\gamma_i| - 1) + 3 - \dim\ V$. If $\pm K_v$ is numerically effective then finitely many such correlators suffice to recover the invariants.

Let me give some notation, we have V and $\Delta_a \in H^{|\Delta_a|}(V)$ and Δ_0 is dual to $[V]$, in general x_a dual to Δ_a and we have the Poincaré form $g_{ab} = (\Delta_a, \Delta_b)$ where $\Delta^a = \sum g^{ab}\Delta_b$. β should be in $H_2(V,\mathbb{Z})$ modulo torsion. It's sometimes useful to have the total correlator $\langle\Delta_1,\dots,\Delta_n\rangle = \sum_\beta\langle\Delta_1,\dots,\Delta_n\rangle_\beta q^\beta$, then $\Phi = \langle e^{\sum x_a\Delta_a}$

We know that $\Delta_a\circ\Delta_b = \sum\partial_a\partial_b\partial_c\Phi\Delta^c$. Then there is a formula

$$\Delta_a\circ\Delta_b = \Delta_a\cup\Delta_b + \sum_{\beta\neq 0}\sum_{c\neq 0}\langle\Delta_a\Delta_b\Delta_c e^{\sum x_k\Delta_k}\rangle_\beta\Delta^c e^{\sum x_k\Delta_k,\beta}q^\beta.$$

The (p,p) part of the quantum cohomology is always endowed with an induced Frobenius structure with flat identity and Euler field. This uses the principle that Gromov-Witten invariants are algebraic and hence of p,p type.

Next, I should talk about tame semisimple, flat metric, and flat coordinates. So first, tame semisimple should be semisimple and endowed with local canonical coordinates u_i, and then you have $u_i(x)\neq u_j(x)$ so that you have something tame.

For quantum cohomology, take $E = \sum(1-\frac{|\Delta_a|}{2})x_a\partial_a + \underbrace{sum_b r_b\partial_b}_{-K_V}$, and the other thing about the flat metric is that a flat metric g satisfying $g(X\circ Y,Z) = g(X,Y\circ Z)$ can be expressed in some canonical coordinates as $g = \sum e_i\eta(du_i)^2, e_i = 2u_i$. For quantum cohomology, this g is just the Poincaré form. We can take η to be the dual coordinate to a point. There is some kind of reconstrcution problem. If the basepoint of your Frobenius manifold M is tame semisimple, then the metric can be uniquely reconstructed from the following data: $\eta_i^0 = (e_i\eta)^0$ and $\eta_{ij}^0 = (e_ie_j\eta)^0$.

Now let's look at the proof of the reconstruction theorem. First, if we know the structure of the whole germ of Frobenius manifolds at 0, which just includes canonical coordiantes and the flat metric $g = \sum e_i\eta(du_i)^2$ plus the expression of (u_i) in terms of some flat coordinates x_i modulo J^2 where J is the ideal of the equaton for $H^2(V)$, and also the correlators with $n\leq 2$, we can reconstruct all of the Gromov-Witten invariants.

Why is this true? The reason is, now we have canonical coordinates and the flat metric, and you want to solve like, some kind of flat coordinates (x_i) from this data, because $\frac{\partial}{\partial x_i}$ is a flat section of the tangent bundle, so it's just a kind of, solve some kind of first order $PDEs$. If you want to determine this, you need to know some initial conditions at this point. If you know that $\vec{u} = \vec{F}(\vec{x}) + O(J^2)$ then you can solve that $d\vec{u} = d\vec{F}(\vec{x}) + O(JdJ)$ where [unintelligible]doesn't come near our [unintelligible].

So we can solve for (x_i) and make the multiplication table as the third derivatives of Φ. The coefficients of the Taylor expansion contain all the Gromov-Witten invariants.

To reconstruct the whole germ of the Frobenius manifold, we have already seen, we have $(u_i^0), \eta_i^0, \eta_{ij}^0$, and to calculate these data, you need (u_i) to be expressed in terms of (x_i) modulo J^2, for the same reason. For the second derivative, we use J^3, because for the second derviative $d\vec{u} = d\vec{F}(x) = 0(JdJ)$.

So now the reconstruction problem is reduced to finding the expressions of (u_i) in terms of (x_i) modulo J^2 plus the correlators with $n\leq 2$. For the first piece, since u_i are the eigenvalues of the canonical Euler field E, if you want to know them modulo J^2 you just need to know the multiplication modulo J^2. So you just need to know Φ modulo J^5, since the multiplication comes from the third derivative. This just comes from the correlators with $n \leq 4$. The information is contained.

3. Second reconstruction theorem

I don't think we defined Gromov-Witten invariants. We have the projective variety X and the moduli space $\overline{\mathscr{M}}_{g,n}(X,\beta) \to X^n$. This is a moduli space of maps $C \to X$ with n points. These maps e_i say just evaluate the map on the ith point. We can take the product ev of these. We also have a map to $\overline{\mathscr{M}}_{g,n}$. We also have do something additional, but let's ignore it.

Let's define $I^{\beta}_{g,n}(\gamma_1,\ldots,\gamma_n)$. Cup classes in X, pull back to $\overline{\mathscr{M}}_{g,n}(X,\beta)$, evaluate against the virtual fundamental class, and push forward along st, this is $st_*([\overline{\mathscr{M}}_{g,n}(X,\beta)]^{vir} \cap ev^*(\gamma_1 \cup \cdots \cup \gamma_n)$.

This definition works for any genus. Let me take $g = 0$ to define Gromov-Witten numbers. Take cohomology classes, then $\langle\gamma_1,\ldots,\gamma_n\rangle_{0,n,\beta}$ will be defined as before, but instead of pushing down along st, integrate.

Today I'll talk about a theorem that I will first formulate and then explain.

Theorem 6. *We have Frobenius manifolds, formal Frobenius manifolds. On the other hand we have, and I will talk about all of the objects in this diagram, we have cyclic $Comm_\infty$-algebras. We also have systems of abstract correlation functions. The theorem says that these objects are in bijective corresponence. There is also cohomological field theories. There is also a bijective correspondence with that. We have an algebraic part, a differential part, and a physical part. The last one is the hardest.*

These are algebraic structures on something. Let's say we are given a $\mathbb{Q}$-algebra k, and let H be a free, finitely generated module over k. So this is $\bigoplus k\Delta_a$. This comes with a pairing, a symmetric nondegenerate pairing $g : H \otimes H \to k$. These are the structures on H. What does it mean to give H the structure of a formal Frobenius manifold? It means the following: We have this basis for H, so pick a dual basis x_a. Let me consider the ring $k[[x_a]]$. To give a structure of a Frobenius manifold means to give a potential ϕ here which makes $H \otimes_k k[[x_a]]$ into a commutative algebra. I need to say that $\Delta_a\Delta_b = \sum_c \phi^c_{ab}\Delta_c$.

Think of a point in the tangent bundle over H, there is a multiplication in this fiber. To pick a point on H means to pick a value for x_a. A point is something like $\sum x_a\Delta_a$. It's the multiplication, not of formal variables but of numbers.

Let me tell you about cyclic Com_∞ algebras. There is an operad Com_∞. If I assume you know operad theory, well, let me make a definition.

Definition 1. *To make H an algebra over the cyclic Com_∞ operad means you provide for every n a function $\circ_n : H^{\otimes n} \to H$ which has the following properties.*

- *S_n-symmetric (∞-commutative [sic])*
- *If you consider the map $(\gamma_1,\ldots,\gamma_{n+1}) \mapsto g((\gamma_1,\ldots,\gamma_n),\gamma_{n+1})$, and*
- *Higher ∞-associativity [sic]. 0 Associativity is*

$$((\alpha,\beta),\gamma) = (\alpha,(\beta,\gamma)).$$

The ∞ associativity sasys for any $\delta_1, \ldots, \delta_m \in H$ and every partition of S into $S_1 \sqcup S_2$, you can put the items in S_1 in the inner product and S_2 in the outer and sum over all possible decompositions.

A system of abstract correlation functions came to us from physics. It means the following. A system of abstract correlation functions means $Y_n : H^{\otimes n} \to k$ for $n \geq 3$ with a splitting axiom.

Let me talk about this equivalence a bit. We have $\circ_n$ on H, how to go to the Y_n? We let $Y_{n+1}(\gamma_1, \ldots, \gamma_{n+1}) = g((\gamma_1, \ldots, \gamma_n), \gamma_{n+1})$. Let's construct a formal Frobenius manifold out of this. We need a potential which is $\sum_{n\geq 3} \frac{1}{n!} Y_n$. The splitting axiom lets us get a potential. How to get from a formal Frobenius manifold back? Write the potential in homogeneous form. This gives you the polynomials Y_n, and take these as abstract correlation functions.

Obviously you can go back and forth between Y_n and $\circ_n$ and you need only check that all properties go back and forth.

Let me mention now what cohomological field theories are. They are the cornerstone, essentially, and this is the reason why there is a "Witten" in Gromov-Witten invariants.

Definition 2. *A cohomological field theory on H is $I_n : H^{\otimes n} \to H^*(\overline{\mathscr{M}}_{0,n})$ satisfying the splitting axom (or its analog) for $n \geq 3$.*

So let's say we have these homological field theories, these maps. You can integrate this over $\overline{\mathscr{M}}_{0,n}$ to get a number, that's how to get from I_n to Y_n. This is tedious but easy. To go from the system of correlation functions to the cohomological field theory is hard. I wanted to give a proof of how to go down but now I don't have time. I want to give an analog. What we don't know is what the field theories and systems of correlation function are. We can think of these as a generalization of Gromov-Witten numbers. The cohomological field theories are a generalization of Gromov-Witten invariants. We don't have the other axioms but it's a generalization. This hard theorem says that if you're given Gromov-Witten numbers then you can reconstruct the invariants in genus zero. Then everything is kind of clear, you have to provide a potential and the coefficients, these numbers, those are Gromov-Witten invariants.