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GRADING EVENT

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1. DECOMPOSITION

Let Q be a finite dimensional $\mathbb{C}$ algebra with a commutative associative multiplication $\circ$ and a unit e. Then every nonzero $x \in Q$ gives rise to a linear map $x\circ : Q \to Q$. We can globally decompose Q into common eigenspaces since it is commutative:

$$Q = \bigoplus_{k=1}^{\ell} Q_k$$

so that for instance $Q_k \circ Q_m = 0$ if $k \neq m$. Then $e = \sum e$. [Write a definition of Q_k.]

Q_k is the generalized eigenspace with respect to λ_k which is a function from $Q \to \mathbb{C}$. This is generalized in the sense that it is $Ker(I - \lambda_k)^n$ for some $n > 0$.

We write $L = \{\lambda_1, \ldots, \lambda_k\}$ which is $Hom_{\mathbb{C}-alg}(Q, \mathbb{C})$. We have a complex structure on L so that $\mathscr{O}_L = Q$ nad $\mathscr{O}_{R,k} \cong Q_k$.

So we can think about a point p, and above it we have an algebra Q. We have this information, we can find L. The concept introduced is that when p varies over a complex manifold M, and we have $Q_{(p)}$, such an algebra, which makes the whole space into a holomorphic vector bundle $\begin{array}{c} Q \\ \downarrow \\ M \end{array}$.

How do we generalize the concept of L to a global thing? We would like to find an object L also over M, so that for each p the fiber is like this discrete set. The answer is that $L = Specan(\mathcal{Q})$, the analytic spectrum of L. We assume each fiber has a multiplication and the unit is a section, where $\mathcal{Q}$ is holomorphic sections of the bundle. In a book, he works in a rather big space Q^*, the dual bundle, to construct, how to construct it. Now $\mathcal{Q}$ is a sheaf but also an $\mathscr{O}_M$ algebra with a multiplication on each fiber $\circ$ and a global unit. Since $\mathcal{Q}$ is an $\mathscr{O}_M$ algebra, we can consider $Sym_{\mathscr{O}_M} \mathcal{Q} \to \mathcal{Q}$ and we give the map $q_1 \otimes \cdots \otimes q_j \mapsto q_1 \circ \cdots \circ q_j$. We also have an ideal J which is the kernel of this map. We will consider in a bigger space Q^* which is also a complex manifold with a complex structure, and we will consider the sheaf $\mathscr{O}_{Q^*}/J$ locally. [We called this the spectral cover in the tangent case. Consider the support. This is our space, $Specan(\mathcal{Q}) = L$.

In general, an analytic spectrum is an object representing a functor. $\mathscr{A}$ is an $\mathscr{O}_M$ algebra. The analytic spectrum satisfies, you have a map $\zeta : Specan(A) \to M$ and $\zeta_*(\mathscr{O}_{Specan(A)}) \cong A$. If there is another map $\varphi : Z \to M$ then a map $Z \to Specan(A)$ above these is the same as $A \to \varphi\mathscr{O}_Z$. [unintelligible]

$\mathcal{Q} = \pi_* \mathscr{O}_L$. So there is one thing. For $p \in M$, the fiber can be composed into eigenspaces. Set $L \cap \pi^{-1}(p) = \{\lambda_1, \ldots, \lambda_\ell\}$. The whole picture is compatible, we have $\mathcal{Q}_p = \bigoplus \mathscr{O}_L, \lambda_k$ with, just, $\mathscr{O}_k(p) = \mathscr{O}_{L,\lambda_k} \otimes_{\mathscr{O}_M} \mathbb{C}$.

So we can extend a neighborhood of the fiber at a point to a little deformation of this.

The next thing is to make an application by the type of decomposition. If n is an integer, the rank of Q over M then $P(n)$ is partitions of integers of n. This has a partially ordered set structure $\gg$. We write $\beta \gg \gamma$ if the pieces of β can be slpit to make γ.

There is a result. Consdier all points of M where the decomposition of $Q(p)$ into $\bigoplus Q_k$ with $P(p) \gg \beta$ has $n = \sum dim_{\mathbb{C}} Q_k$, the result is, this is an analytic subset in M. The next result is that there exists β_0 so that the $p \in M$ ith $P(p) = \beta_0$ this is an open set. The generic semisimple case has $\beta_0 = (1, \ldots, 1)$.

2. Sam Gunningham

Decomposition of F-manifolds.

This is going to be an extension of the previous talk. In the previous talk, we had an algebra Q. At the beginning, it was finite dimensional and we decomposed it into a direct sum of generalized eigenspaces, and extended that to the case of a vector bundle with multiplication, and we got a decomposition in the same way. Taking this one step further, if $(M, \circ, e)$ is an F-manifold then $\mathscr{T}_M$ has a multiplication $\circ$ so we get a decomposition locally of $\mathscr{T}_M$ into $\bigoplus \mathscr{T}_k$. Here in this talk, we'll show that we can integrate this decomposition to a decomposition of manifolds (locally). You have $M = \prod M_k$ so that the tangent bundle of a part M_k is $(T_p M)_k$.

It'll amount to solving PDEs to give us a picture along the base. The F identity will give us the integrability conditions. Maybe first I'd just like to do an example to see exactly how this is, what the difference is between this result and the previous one. If $M = \mathbb{C}^2$ with coordinates t_1 and t_2 We'll have $\partial_1 = e$ and $\partial_2 \circ \partial_2 = t_2 \partial_2$. You can check the F-identity and also that when $t_2 \neq 0$ this is semisimple and then we have idempotents, projectors $e_1 = \frac{1}{2}\partial_1 \div \frac{1}{2\sqrt{t_2}}\partial_2$ and $e_2 = \frac{1}{2}\partial_2 + \frac{1}{2\sqrt{t_2}}$.

We'd like to find local (canonical coordinates) on M so that $\frac{\partial}{\partial u_i} = e_i$. This gives the sysstem of PDEs:

$$du_1 = dt_1 - \sqrt{t_2} dt_2, \qquad du_2 = dt_1 + \sqrt{t_2} dt_2.$$

We solve these and get at $p = (a, b)$ the equations $u_1 = t_1 - \frac{2}{3} t^3 2_2 - (a - \frac{2}{3} b^{\frac{3}{2}})$ and $u_2 = t_1 - \frac{2}{3} t^3 2_2 - (a \frac{2}{3} b^{\frac{3}{2}})$ and

The outline: we have a decomposition (everything will be local) $\mathscr{T}_M / \bigoplus \mathscr{T}_k$ and the first step is to show that the F-identity impries the following integrability condition: $[\mathscr{T}_i, \mathscr{T}_j] \subset \mathscr{T}_i + \mathscr{T}_j$. How do we use this? The proof of this claim is basically sort of a manipulation of the F-identity. You can play around with some symbols. The harder bit of the theorem is wrapped up in this: (Frobenius integrability theorem) which says that if $\mathcal{E} \subset \mathscr{T}_M$ is a subbundle of $\mathscr{T}_M$ and $[\mathcal{E}, \mathcal{E}] \subset \mathcal{E}$ then we can find a submersion $f : M \to \mathscr{C}^{n-r}$ (where r is the rank). Tangent vectors along the fibers span $\mathcal{E}$. We can define $\mathcal{E}_i$ to be $\bigoplus \mathscr{T}_k$. Then the conditions I'm erasing give us that these subbundles are integral, so we get $f_i : M \to \mathbb{C}^{n_i}$, with $n_i = rk(\mathscr{T}_i)$. This gives us the decomposition we want. Set M_i to be $\{f_j = 0 | j \neq i\}$ so that when we put all the fs together we get a decomposition. There's still something that needs to be checked. We've got a decomposition so that the tangent space is the eigenspace $\mathscr{T}_i$ but we still need to check that the multiplication descends to one defined only on M_i. It could be that multiplying lifts of two vectors in $\mathscr{T} M_i$ could vary in the fibers. This is just another application of the F-identity.

This is basically how the proof works. I could go into details of one of these lemmas or say something about Euler fields. It's also probably useful to note that this decomposition is compatible with Euler fields in the following sense: If $M = \prod M_i$ as F-manifolds (implicit is that the product of F-manifolds is one) and E is an Euler field of some weight d on M, then E_i (the projection of E) is an Euler field on M_i. This is proved in a very similar way to the other facts. You'll use the definition of an Euler field. I can show that or finish.

[This part of the theory shows that the usual operation of taking the direct sum or product of commutative algebras is locally globalized to F-manifolds if you have an F-identity, and what is interesting is that on commutative algebras you have a tensor product, which distributes over the direct sum. It's not at all obvious that you can extend the tensor product. You can, this is the Kunneth formula for quantum cohomology. It extends this formula. If you look at F-manifolds from singularities. If you have an unfoldinig that is isolated. If you have one $f(x)$ and another $g(y)$, you take some sort of sum.]

[I missed Ian's talk.]