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Recall that we are working with $I^V_{g,n,\beta} : H^*(V)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$, and the main diagram that is needed in order to define these maps is like this:

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{g,n}(V,\beta) & \\ {}^{ev}\swarrow & & \searrow^{q} \\ V^n & & \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(pt) \end{array}$$

We pass to classes of algebraic cycles and treat this as a correspondence on the product. The most difficult and nontrivial part of this story is the construction of the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{vir} \in A_*(\cdots)$. This requires a notion of stability, and this has become a variable. Then different things in this change.

The philosophy is that of a moduli space of "something," we let $\overline{\mathcal{M}}$ lead to $\overline{\mathcal{M}}(S)$ where S is a variable scheme, the class of families of something over the base S which should be a category. We can sometimes move along this arrow in the other direction to use this to define $\overline{\mathcal{M}}$. This philosophy takes into account two contradictory desires. What families do you consider? The minimal condition is that of flatness. The basic situation is when you have a morphism of two affine schemes, that corresponds to a morphism of commutative rings in the opposite direction. Flatness means that when you lift a three-term exact sequence you get a three-term exact sequence. Intuitively, flatness means continuity in the sens of algebraic geometry. The next condition is, you are considering objects of general type, this was always the concern, starting in Euclidean geometry. You imagine it has three different vertices and three different lines. But what if all the vertices are on the same line, or what if they all three coincide. In the very degenerate case, then the side that connects the two vertices is any line passing through them. This is a prototype of a trouble in algebraic geometry. When you pass from flat families to a "degenerate case" then you have an infinity of possibilities. On the level of moduli spaces, a triangle generically is given by $E^2 \times E^2 \times E^2$, but when you sit on the diagonal, you have infinitely many choices. One obvious stability condition is to fix a line, and declare that the side should be parallel to a given line. The stability conditions could be done in a number of ways, this kind is in bijection with lines. Similarly here, you glue your infinity of possibilities one and the same thing, but you do not glue the points, and lose Hausdorffness. This would be true if this is all you care about, flatness. Adding stability conditions, you cut down on the points you allow.

Let's do this in our example. We will need stable maps, and from that stable curves. We'll work in the other direction. The lower level is Gromov-Witten invariants of arbitrary genus or genus zero, and above is motivic Gromov-Witten.

Now, stability. We will use the concrete (most popular) stability conditions for $(C, x_1, \ldots, x_n)$ and maps from this into V.

Definition 1. *A* prestable curve *over a base T is a morphism $\pi : C \to T$ whose geometric fibers are curves: 1-dimensional, reduced schemes, with a strong restriction to singularities which are only double points and only with distinct tangents. Such a curve has genus over a point, which is $g(C_t) = \dim H^1(C_t, \mathscr{O}_{C_t})$. Usually one defines genus in a dual way, taking H^0 of Ω. Singularities introduce new structure, and this is the shortest definition. Labeled points are sections $T \to C$, labeled by a finite set. Then the simplest definition of stability is, a prestable curve is* stable *if the sections are pairwise disjoint and do not pass through singularities and the normalization of any irreducible component contains at least 3 points if $g = 0$ and at least 1 if $g = 1$.*

This is easily seen to be equivalent to the fact that the automorphisms of any fiber, fixing marked points, is finite. Although they are easily equivalent, intuitively, the automorphism condition is the important one.

Why should this be the important condition? Let us return to our philosophy. For any S I have $\overline{\mathcal{M}}(S)$, flat families satisfying stability conditions. How can I reconstruct $\overline{\mathcal{M}}$ from $\overline{\mathcal{M}}(S)$ and additionally from the functor's action on the morphisms, $S \to S'$ leads to $\overline{\mathcal{M}}(S) \to \overline{\mathcal{M}}(S')$.

First, let's translate some of this data? I'm assuming now that $\overline{\mathcal{M}}$ does exist and is a scheme. If $\overline{\mathcal{M}}$ is a base of a universal family, how? I can take the identity morphism of $\overline{\mathcal{M}}$, since it is a scheme, so the identity morphism should correspond to a family $\tilde{C}$ over $\overline{\mathcal{M}}$. Suppose I start with base S. Then S carries its own family C over S. This should be obtained by an arrow $C \to \tilde{C}$. Each family $C \to S$ should uniquely define the arrows $C \to \tilde{C}$ and $S \to M$.

How do you relate this to automorphism? Imagine $\overline{\mathcal{M}}$ is a point. If I want $C \to C$ to be unique then there must be no automorphism of C. So if I want my moduli space to be represented by a scheme, then there should be trivial automorphism groups. Let's temporarily call this a *very stable* family. For example $g = 0$ and $n \geq 3$, we get $\overline{\mathcal{M}}_{0,n}$. For $g = 1$, if I take $n = 1$, there are automorphisms everywhere, and there are two points with automorphisms $\mathbb{Z}_4$ and $\mathbb{Z}_6$. Whatever you do you cannot get a moduli space for genus one as a scheme, and this is the simplest example of a stack. We have a "crude moduli space" where we let each object enter only once. You should have instead a quotient of this object, modulo a trivial action of $\mathbb{Z}_2$. This is well-known. There are two ways to include these in algebraic geometry, going to noncommutative geometry and going through stacks. You will see right away, you see the first Deligne-Mumford stacks. In algebraic geometry you might have an infinite automorphism group, like $\mathbb{P}^1$. The moduli space is a point modulo $PGL(2)$ so is $BPGL(2)$. We don't have that here. You restrict yourself and do not get anything worse than a DM stack. For $g \geq 2$ you don't need a marked point, but nevertheless the groups are there.

It's silly to prove that you get a scheme directly, it makes more sense to go through stacks and show that a stack with trivial group is a scheme.

Now let's pass through maps. I'm considering not just individual maps but families of maps. I think it's more or less clear what is a family of maps. This time, prestable is very much essential. Then I have a map of $C \to T$ into V. If this is flat, then everything I declare to be flat as well. How do I change the stability condition? We have Ω_C, the sheaf of 1-forms, and the "dualizing sheaf" ω_C, and there is a morphism $\Omega_C \to \omega_C$. In the prestable case this allows a very precise explicit description of the dualizing sheaf via, not Ω_C but the sheaf of one-forms on the normalization of C, $\Omega_{\tilde{C}}$ where $\tilde{C} \to C$ is normalization, where you desingularize.

What happens now, if I take my dualizing sheaf and want to understand its sections over U, then $\Gamma(U,\omega)$ is the sections of the preimage of Ω with [unintelligible]singularities, define D to be the sum of all points that project onto singular points. So sections $\Gamma(f^{-1}(U),\Omega_{\tilde{C}}(\log D)$, which allows logarithmic singularities. There is an additional condition, that if $x \neq y$ and $f(x) = f(y)$ then $res_x\nu + res_y\nu = 0$. So this should take into account what gets glued together. The simplest explanation for why, not a proof, is that when I have such a singular point and I take the completion of the local ring of this curve here, I have $\urcorner[[u,v]]/(uv)$. Since I am factorizing, this means $d(uv) = 0$, which is $du \cdot v + u \cdot v$ which is the same as $\frac{du}{u} = -\frac{dv}{v}$. This is too strong, but the residues remain the same.

Now we know how to characterize, and the name of the "dualizing" sheaf comes from the fact that we have the canonical duality $H^1(C,\mathscr{F}) \times Hom_{\mathscr{O}_C}(\mathscr{F},\omega_C) \to H^1(C,\omega_C) = \urcorner$, and this is a perfect pairing.

Instead of using differential forms, you should use this dualizing sheaf.

Then we need one more reformulation of the stability condition for an individual curve. A prestable curve is stable if and only if when I allow for $\omega_c(x + \cdots + x_r)$ with labelled points, then it becomes ample and the points do not intersect. Then $[\cdots]^{\otimes 3}$ is very ample.

This immediately generalizes to families, just because unlike differential forms, which fail to glue together well if they're singular. But here you get a relative dualizing sheaf $\omega_{C/T}$.

Second, it has a natural generalization to maps. If I have a prestable $C \to T$ with sections x_n and a map $f \to V$, I can ask additionally, V will be a projective smooth variety with ample sheaf M. Instead of the ampleness condition, require that $\omega_{C/T}(\sum x_i) \otimes f^*(M)^{\otimes 3}$ is ample. If M is not a point there is another circumstance to allow you to stabilize.

Let me illustrate in genus zero. You might have an embedding of a curve of genus zero $\mathbb{P}^1 \to V$. If you forget V, $\mathbb{P}^1$ is unstable, but the inverse image of an ample divisor is ample as well. So this shows one very important property, which is sometimes overlooked: there are unstable cases in $\overline{\mathcal{M}}_{g,n}$ where this doesn't exist as a Deligne-Mumford stack because the automorphism groups are infinite. So your diagram becomes incomplete. This shows you there can be trouble with these invariants having 2, 1, or 0 arguments, or genus 1 and 0 arguments. To define them you should work entirely in $\overline{\mathcal{M}}_{g,n}(V,\beta)$, and add additional arguments like the identity class on V. This is very important but more important is that one should consider some way out to put something in as $\overline{\mathcal{M}}_{g,n}$ or see what you can do with Artin stacks, quotients by algebraic groups with dimension more than 0. It would be nice to consider $\overline{\mathcal{M}}_{0,2}$ which should be BG_m. I'm not sure whether we can input this story. I was speaking of it with a German, maybe Polish mathematician in Bonn and I think he has several papers saying what happens when one sets here an Artin stack.