MATH PHYSICS SEMINAR, GLOBAL β-FUNCTIONS UNDER ζ-FUNCTION REGULARIZATION SUSAMA AGARWALA

GABRIEL C. DRUMMOND-COLE

Thank you for inviting me. The β function for a scalar field theory is understood in flat space time, but I want to think of it globally over a manifold. One of the keys to doing this is that Connes and Marcolli found a way to represent β-functions as geometric objects. So I want to see what implications this has for conformal field theories.

A quantum field theory is defined by a Lagrangian density, such as

$$\frac{1}{2}\phi(x)(\Box + m^2)\phi(x) + \lambda\phi^3(x)$$

This gives me probability amblitudes for $\phi : e^{i\int \mathcal{L}}$. Perturbative expansion gives Green's functions. These have intgrals that are not well-defined, such as

$$K_\Gamma(a,b) = \iint G(a,x)G(x,y)G(x,y)G(y,b)dxdy$$

The usual solution is to introduce regularization parameters $\mathcal{L} \to \mathcal{L}(z)$. The problem is that this is no longer scale invariant, and you get a renormalization group. You need to keep track of your scale, so you get a second parameter $\mathcal{L} \to \mathcal{L}(t,z)$.

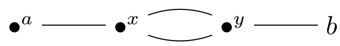
Now you can renormalize and try to get a number out: $\mathcal{L}(z,t) \to \mathcal{L}(t) = \mathcal{L}_{ct}(t) + \mathcal{L}_{fin}(t)$. So the renormalization group equations $\frac{\partial}{\partial t}\mathcal{L}_{ct}(t) = 0$ are equivalent to the condition that physicists call locality.

So the easiest one to solve is $\beta(\lambda) = t\frac{d}{dt}(\lambda)$.

[Some quick examples, like QED and QCD and general Yang-Mills]

Now I'll talk about a geometric picture. I'll define a scalar field theory over a compact manifold with no boundary with a Riemannian metric (M,g). Instead of $\Box + m^2$ I'll take $\Delta_M = \Delta_g - m^2$. We can explicitly write the Green's functions $G_M(x,y) = \sum \frac{\phi_i(x)\phi_i(y)}{\lambda_i}$. I'll also do my calculations in configuration space.

Let me talk about the integrals associated to these Feynman diagrams. I want a Schwartz kernel. So for instance I may look at this simplest diagram



and my integrals will not converge, badly.

One way of fixing this is to raise the Laplacian to a complex power $1+z$, so then we get a trace class operator for certain values of z.

Once I've done that, the regularized Feynman rules for a graph Γ with E_Γ external legs is a Schwartz kernel $K_\Gamma^{reg} \in \mathcal{D}'(M^{E_\Gamma})z$.

The proof involves the Mellin transform of the heat operator. The Green's function is meromorphic in z with simple poles at $k - \frac{n}{2} - 1$. My manifolds should be dimension four or six.

Combining everything I get a definition for K_Γ^{reg}. This has been done over flat spacetime. This is the exact same story for curved spacetime. That's the geometric side. There is also the renormalization bundle side. The story here, my manifold is $\mathbb{R}^6$ with a scalar QFT. The regularization method is dimensional regularization. Now I integrate over a non-integer dimensional space $\mathbb{R}^{6V+\epsilon}$. We do this kind of thing all the time when we take Mellin transforms. Just, before you throw a fit, think of it that way.

Then there's this Hopf algebra of Feynman diagrams. You get as an underlying algebra, the complexes of 2 connected graphs (if you remove an edge it's still connected). Your addition is formal sums. The multiplication is disjoint union. The interesting part is the comultiplication. So $\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum \gamma \otimes \Gamma//\gamma$.

So loops create divergences. I'm not interested in the big loop [picture] because it has four edges coming out. I'm only interested in ones that could still be in my theory when I contract.

I take γ a subdivergence out, and then contract it within Γ to get $\Gamma//\gamma$.

The term $1 \otimes \Gamma$ is the trivial cut; the term $\Gamma \otimes 1$ is from cutting out the whole graph. Then I take all of the other such divergences (subgraphs). It's interesting, this happens to be coassociative. The counit takes the identity to 1 in $\mathbb{C}$. The antipodal map is recursive. It comes from $m(1 \otimes S)\Delta(x) = \eta\epsilon(x)$. You can force the definition out of this.

This is graded by the first Betti number of the graph. The restricted dual Hopf algebra (the dual of the graded parts) is generated by the primitive elements δ_Γ. The multiplication is the convolution product $\phi \star \phi' = m(\phi \otimes \phi')\Delta$ and the comultiplication is defined by $\Delta(\delta_\Gamma) = 1 \otimes \delta_\Gamma + \delta_\Gamma \otimes 1$. The antipode of ϕ is $\phi^{\star -1}$.

A cocommutative connected Hopf algebra tells you that this is the universal enveloping algebra of a Lie algebra, which happens to be the Lie algebra of a Lie group G. So I can write the Birkhoff decomposition $\phi(z)\Gamma = \phi_-^{\star -1}(z) \star \phi_+(z)(\Gamma)$. If I create an infinitessimal disk Δ in $\mathbb{C}$ and a bundle $\Delta \times G$ over it. Sections of this is $\phi(z)$. Here $\Delta = \underbrace{\mathit{Spec}\ \mathbb{C}[z^{-1}, z]]}_{A}$ (formal power series in z and meromorphic functions in z^{-1}). So $\phi : \mathit{Spec}\ A \to \mathit{Spec}\ H$. So the expressions for ϕ_- and ϕ_+ are the counterterm and the finite part expressions, so that's really cool.

Now I need to incorporate the renormalization group. Then Connes and Marcolli showed that if the section satisfies the locality condition $\frac{d}{dt}(e^{tY}\phi(z))_- = 0$ then there is a flat connection on an associated bundle which incorporates the renormalization group (P). The $\beta(\phi)$ function ends up being the residue of $\phi_-(z)$. The connection, the flat connection ends up being $\phi^{-1}d\phi$. So if they have the same pole part they'll define the same connection.

[What about non-renormalizable?]

You need an infinite family of counterterms. So you don't have a well-defined Hopf algebra.

Now I want to put the two sides together. One of the key points is that in order to define a β function you want the sections to satisfy this locality principle. I claim that I can do the same story with ζ-function regularization.

[Many equations, comparing K_Γ^{reg} and $K_\Gamma^{dim\ reg}$.]

So this doesn't change my pole structure. In fact, over flat spacetime the β functions are exactly the same.

So what does this say over a curved background manifold? Then I can construct this bundle and the new section that I'm interested in, and the β function for a ζ-regularized theory is defined by a flat connection.

That's the story for nonconformal field theories. What about the conformal case? The regularized Lagrangian density $\mathcal{L}(z)$ is not scale invariant. We have two possible solutions: scale the metric globally (easy) or conformally (hard). People have made a lot of progress looking at a manifold with patches, but conformally it's hard.

If I have a frame bundle on M^n, I can define a line bundle on it using the determinant. Sections of this bundle will be r-densities. If I have a smooth function on my manifold, I can trivialize by picking a metric. This lets me define a Banach norm on the r-densities, so a Lebesgue space $\mathbb{L}(r)$. I want the one that's a Hilbert space. There is a conformally invariant Lapalacian. This only works for scalar field theories. [Equation]. This goes out of our Hilbert space. So we can do something to get an operator Y_g that stays in the Hilbert space, and then I'll regularize by substitution. I can expand this as a Taylor series in f and I get $e^{-2fz}\tilde{Y}_g(z)$. Now with the regularized operator, I have a conformal field theory.

I can get a conformal β-function by a connection on the bundle $G \times \mathcal{B}_M \times_{GL_n(\mathbb{C})} Frame\ M$ over $\Delta^* \times M \times_{GL_n(\mathbb{C})} \mathbb{C}^\times(1)$.