COLLOQUIUM
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MAPS INTO AND OUT OF AUTOMORPHISM GROUPS OF FREE GROUPS

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It's always a pleasure to come to Chicago. I'm happy to be here. I found this picture on your webpage [laughter.] There are strong analogies between outer automorphism groups of free groups and mapping class groups and lattices. I'm going to ignore mapping class groups, although many of the things I'm going to say have to do with those too.

Here's an Escher print of a lattice, but I'm thinking of irreducible lattices in higher rank semisimple Lie groups. It's a discrete subgroup Λ of G so that G/Λ has finite volume. The classical example is $\Lambda = SL(n, \mathbb{Z}) \subset SL(n, \mathbb{R})$ which has a finite volume noncompact quotient. Another example is $\Lambda = \pi_1(M_3)$ for a hyperbolic manifold. A couple of nonexamples are $\mathbb{Z}^n$ in $\mathbb{R}^n$, which is reducible, and also not $\Gamma \subset PSL(2, \mathbb{R})$ which are only rank one, so for me n is higher.

The other thing is automorphisms of free groups. So for example ρ_{ij} takes x_i to $x_i x_j$ and preserves the other generators. There's also a left multiplication λ_{ij} which does the same kind of thing, and the permutation e_{ij}. Nielsen proved that these three types generate the automorphisms. This was in the 1920s and 1930s. Other people interested in this were J. H. C. Whitehead and W. Magnus. They used algebraic and topological methods, but later progress was slow. The free group is the fundamental group of a handlebody, or doubled handlebody, and these induce homeomorphisms. Then after the 30s, people had a hard time proving things about these groups. John Stallings introduced new topological techniques. He said that you should understand graphs. To understand automorphisms of free groups, you should understand homotopy equivalences of graphs. More tools were introduced by Bestvina-Handel, Culler-V., motivated by Thurston and Gromov. There has been an explosion of activity since then. I will show you a diagram. You have an abelianization $F_n \to \mathbb{Z}^n$, and an automorphism induces an automorphism, so there is a map $\mathrm{Aut}F_n \to \mathrm{Aut}\mathbb{Z}^n$ which is $GL(n, \mathbb{Z})$. There is a kernel IA_n, which always includes the inner automorphisms, and it's really a mysterious group in general. I'll call $S\mathrm{Aut}F_n$ the preimage of $SL(n, \mathbb{Z})$ in $GL(n, \mathbb{Z})$. So it would be obvious that there was a relationship if $\mathrm{Aut}F_n$ was a lattice, but a normal subgroup of a higher rank lattice is central or of finite index. But IA_n is not central and does not have finite index for $n > 2$. They also are too big to be rank one lattices. So but there should be an analogy.

Let me convince you that they're really not lattices. A higher rank lattice Γ, a map to $Out(F_n)$ has finite image. This uses the work of many people. [Names]

So they don't even contain lattices. The same theorem for surface mapping class groups is also true.

So I claimed that they were like lattices. What do they share? They are finitely generated, and indeed finitely presented. But a group that is finitely generated, its homology is finitely generated, and finitely presented means that the second homology is finitely generated. But these are finitely generated homology in all dimensions, and rationally finite dimensional There are finitely many conjugacy classes of [unintelligible]. So these are finite.

The subgroup structure, every subgroup of a lattice has a nonAbelian free subgroup or is solvable. For $Out(F_n)$, the solvable subgroups are Abelian.

There are more cohomological properties. There is a duality between homology and cohomology, and lattices sometimes comes in sequences, and the inclusions $\Lambda_n \to \Lambda_{n+1}$ induce an isomoprhism on homology for big enough n, which is also true for automorphisms of free gorups.

Lattices act on an interesting contractible space G/K with finite volume quotient. For $Out(F_n)$ this is Outer space. These actions can be used to establish many of the common properties.

Here's a picture of outer space, for $n = 2$. [Picture]

Notice that outer space isn't a manifold, it's a cell-complex, it has these fins sticking up. We have a duality between homology and cohomology. That's what was used in the case of lattices. Here it's harder to do because manifolds have Poincaré duality, these don't.

The topic I wanted to concentrate on is rigidity properties. So for example, Margulis superrigidity says that a homomorphism of (higher rank irreducible in connected semisimple Lie groups) lattices extends to a homomorphism of the ambient groups. Then for example a map $SL(n,\mathbb{Z}) \to SL(m,\mathbb{Z})$ extends to $SL(n,\mathbb{R}) \to SL(m,\mathbb{R})$. So if $n > m$ then there are no representations other than the trivial one.

Prasad noticed that an injection $\Lambda \to \Lambda$ is surjective, and the outer automorphisms of a lattice is finite.

What about $\mathrm{Aut}F_n$ and $Out\ F_n$? Formanek and then Bridson and I proved that automorphisms of $Out\ F_n$ are inner. An injective map from $Out\ F_n$ to itself is surjective, and a surjective map is injective. So then if $m < n$ then a map $S\mathrm{Aut}F_n \to S\mathrm{Aut}F_m$ or $SOut\ F_n \to SOut\ F_m$ is trivial. (without the S it could be $\mathbb{Z}_2$).

I wanted to talk about the question, what if $m > n$? There's an obvious inclusion of the automorphisms where you think of an automorphism as acting on the first n generators. But inner automorphisms are no longer inner. So in particular this does not induce a map on the outer automorphism groups. Are there any interesting maps at all between the outer automorphism groups for $n < m$? So Khramtsov showed there are no embeddings if $m = n + 1$. He proved this in Russian, so it took us a while.

We showed, if $m < 2n - 1$, any map form $Out\ F_n$ to $Out\ F_m$ has finite image. If n is even, and bigger than two, $m \leq 2n$.

[The outer homology groups stabilize, although that's funny because there are no inclusion maps.]

Khramtsov did construct an embedding $Out\ F_2 \to Out\ F_4$. Later this was done for $m = r^n(n-1)+1$ with r odd and prime to $n-1$. They conjectured that $Out\ F_n$ embeds in $Out\ F_{2n}$, but our theorem shows this is false.

So how do you do this? Take a characteristic subgroup H (take a finite index subgroup, index k, and intersect all of them). It's an exercise that it's finite index. So $H \cong F_{k(n-1)+1}$. This is an exercise in Euler characteristic. Take a k-fold cover of a rose with n petals.

We get a restriction map $\text{Aut}F_n \to \text{Aut } H$. This has to be injective since H is finite index, and sends H (viewed as an inner automorphism subgroup) to H so it induces an injection $\text{Aut } F_n/H \to \text{Aut } H/H = Out(H)$. The domain is not $Out\ F_n$. However, I could mod out the rest of them and get a map $\text{Aut } F_n/H \to Out\ F_n$. If I can split that map we can find such an embedding. So they found a splitting for some H containing the commutator $[F_n, F_n]$. Martin and I proved that if H contains the commutator, then q splits if and only if $H = (F_n)^r[F_n, F_n]$ with $(r, n-1) = 1$. This includes their example, but also includes even r.

This is a technical looking statement, but this says when you'll get a splitting from their method, and that we can't hope for others with this method. I want to say a couple of remarks. First of all, we didn't believe this. I want to talk about the proof of the first theorem, but point out first the consequence for the cohomology of $Out\ F_n$.

I have a short exact sequence with a kernel which is a free group, and sometimes it doesn't split. So the obstruction comes from $H^2(Out\ F_n, H^1\ F_n)$. Here $Out\ F_n$ gives a matrix which acts on $H^1\ F_n = \mathbb{Z}^n$. However, $H^2(Out\ F_n, H^1\ F_n) = 0$ for large enough n. But here $Out\ F_n$ is acting by taking v not to Av but to the ${}^tA^{-1}v$, which gives different cohomology.

Let's go back to theorem one. We want to show that $Out\ F_n \to Out\ F_m$ has finite image if $m < 2n-1$, $m \neq n$. I'll start with $m < n$.

I claim that $Out\ F_n$ contains an alternating group A_{n+1} and if $m < n$, $Out\ F_m$ does not contain an A_{n+1}. So any map $SOut\ F_n \to SOut\ F_m$ must send A_{n+1} to 1. Then $SOut\ F_n$ is generated by ρ_{ij} and λ_{ij} (the determinant one automorphisms don't need the permutations). Then $\sigma\rho_{ij}\sigma^{-1} = \rho_{\sigma(i)\sigma(j)}$ and $[\rho_{ij}, \rho_{jk}] = \rho_{ik}$. Then take a σ in A_{n+1} with $\sigma(i) = j$ and $\sigma(j) = k$ so $\sigma\rho_{ij}\sigma^{-1} = \rho_{jk}$. Then $\rho_{ij} = [\rho_{ij}, \sigma\rho_{ij}\sigma^{-1}] \mapsto [f(\rho_{ij}), f(\rho_{ij})] = 1$.

We needed that $Out\ F_n$ contains this alternating group and $Out\ F_m$ does not contain A_{n+1} for $m < n$.

Here's a graph with fundamental group F_n. The alternating group permutes the edges, which induces an automorphism of the free group. So we say that the graph *realizes* A_{n+1}. Only finite groups are automorphism groups of graphs.

For the second part, we need a nice theorem, proved independently by three researchers, who showed that every finite subgruop of $Out\ F_n$ is realized on some connected graph with no separating edges, univalent, or bivalent vertices. A quick Euler characteristic computation shows that such a graph of rank n has fewer than $2n-2$ vertices and $3n-3$ edges. Suppose you had A_{n+1} inside $Out\ F_m$. Then you'd have a graph of rank m with this A_{n+1} in its automorphisms. But then the orbits have to have size 1 or $n+1$ because otherwise there'd be a map to a smaller symmetric group with a kernel. If all the vertex orbits are trivial, X must contain a cage or a rose. You'd have a subgraph of rank n for a cage and $n+1$ for a rose. There has to be a big vertex orbit. Then X contains, if there's a loop at one there's a loop at all and you have rank $n+1$. If there's an edge between two of those, then, A_{n+1} is doubly transitive, and you'd need a complete subgraph, rank $n(n-1)/2$ So then there are 3 different edge orbits, but there are only $3n-3$ edges.

That's the end of the proof if $n > m$. But if $n < m \leq 2n$ there are graphs realizing A_{n+1}. The idea is that there are such graphs, but they're pretty rare. You can classify them for key finite subgroups of *Out* F_n. Then you look at these. You have a graph realizing automorphisms. You look at invariants associated to the actions that could exist. Then you show that if two graphs realize two groups, then both graphs realize their intersection. Then you show that none of the sets of homomorphisms $G \to Out\ F_m$ extends to *Out* F_n.

[Picture]

Part of the proofs of this theorem is classifying graphs. These are all of them.

Some remarks. The first embedding given by our embedding theorem is $m = 2^n(n-1)+1$, which is exponential in n. So the bound $m < 2n-1$ is far from optimal. A student of Bridson has improved this bound. Last time I checked I think it was up to $3n$. The bound is better if m is odd, because there are more possible actions on the graphs, which makes them harder to analyze.

The last thing I wanted to do is talk about the other half of the theorem, when this map splits.

Here's a graph with fundamental group F_n, and there's a cyclic subgroup here in *Out* F_n, and no matter how you try to lift that, it would have infinite order, so this map can't split unless $(r, n-1) = 1$. The appropriate cover of the graph looks like [Picture]. You'll only get something homotopic to a deck transformation of that cover if you have this relative primeness. So this is changing an algebraic question to a geometric question about graphs. I'm going to end there with Escher's picture again.

[Anything known about finite index subgroups?]

Here's something known. A finite index subgroup (characteristic) you get a map like this, you can embed a finite index subgroup of *Out* F_n, there exists one that can embed in *Out* H. The kernel of the map to $SL(n, \mathbb{Z})$ is torsion free.

[What about IA_n?] Are there embeddings of IA_n into IA_{n+1}? I don't know.

[You mentioned a similarity between outer automorphisms and lattices. There is a theorem or conjecture that higher rank lattices do not act faithfully. Does it come from, is it a similarity like this?]

The proof I gave you also proves that *Out* F_n can't act on S^1 by homeomorphisms. No one understands finite index subgroups. In fact $OutF_n$ cannot act on S^m for $m < n$.

[Does the similarity come from something?]

The proof of the Tits alternative is very different for the two cases.

[[unintelligible]. Can I think of this as coming from something changing a graph to another graph?] You're finding a cover of that graph. You can lift homeomorphisms of the graph to homeomorphisms of the cover if it's a characteristic cover.