GEOMETRY/PHYSICS SEMINAR
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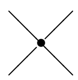
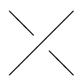
1. Pretalk on finite type knot invariants and generalizations

Thanks for the invitation and thanks for coming out so early in the morning relative to most seminar times. Rigorously, a knot is an embedding, for this talk, of S^1 into $\mathbb{R}^3$, up to isotopy. That's a path in the space of embeddings. Think of this as being able to wiggle around without passing through itself. A knot invariant is a function on isotopy classes of knots. This might land in, for most of this talk it will be in $\mathbb{R}$. In general it could land in any field or ring. About half of this talk, I'll give a combinatorial description of finite type knot invariants. Then I'll talk about the Bott-Taubes configuration space integrals. Perhaps at the very end I'll talk about other generalizations.

A *finite type knot invariant* or Vassiliev invariant (these were constructed around 1990 and Bar-Natan has a good paper in the early 90s, which gives the combinatorial description) is, well, start with any knot invariant. You can extend it to knots with a finite number n of transverse self-intersections via the following *skein relation*:

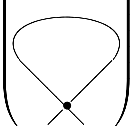
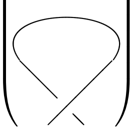
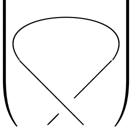
$v^{(n)}\Bigg($  $\Bigg) = v^{(n-1)}\Bigg($  $\Bigg) - v^{(n-1)}\Bigg($  $\Bigg)$.

Definition 1. *An invariant is finite type of type m if it vanishes on singular knots with more than m self-intersections. We'll call V_m the vector space of type m invariants and then we have a filtration*

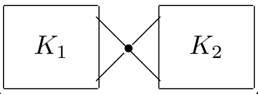
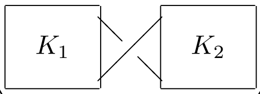
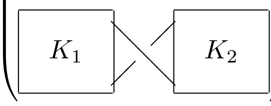
$$V_0 \subset V_1 \subset \cdots$$

A useful analogy of Bar-Natan is to think of the extension to knots with m double points as an mth derivative of the invariant.

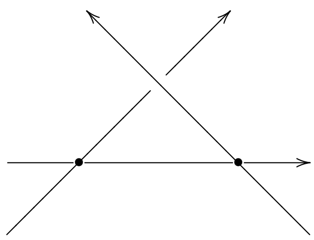
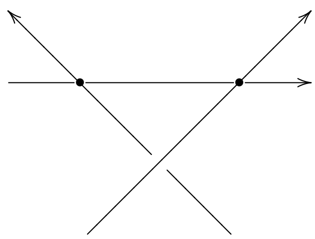
Observe that any finite type knot invariant satisfies:

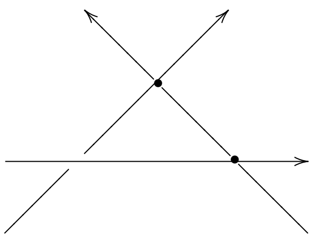
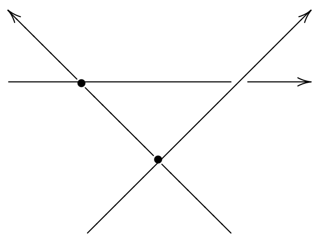
(one term relation) $v\Bigg($  $\Bigg) = v\Bigg($  $\Bigg) - v\Bigg($  $\Bigg) = 0.$

There's also the generalized one which has to do with connect sums, so
(generalized one term relation)

v  $= v$  $- v$  $= 0.$

Similarly, the skein relation implies:

v  $- v$ 

$+v$  $- v$  $= 0$

One implication is that if $v \in V_m$ and K has m double-points then any crossing change to K leaves $v(K)$ unchanged.

So immediately, $V_0 = \mathbb{R}$ and $V_1 = \mathbb{R}$ since any knot with one double point can be changed to one particular singular knot. So similarly, $v(K)$ is determined by the placement of the double points in the domain S^1.

So a chord diagram is a circle with m pairs of points joined by chords up to diffeomorphisms of S^1. This is a purely combinatorial or topological object. We don't care the lengths or how far apart these are or anything like that. Let's define further Δ_m as the vector space of chord diagrams of degree m where we formally add and multiply them. For example, Δ_2 is two dimensional, spanned by an "X" and a "double theta."

Another example, slightly less trivial, if you look at Δ_3, this is generated by a pie (six slices), the basketball, the AT&T logo, the frowning ninja turtle, and the monkey saddle.

To any $v \in V_m$, we can associate a functional $w(v) \in \Delta_m^*$. What you do do? You associate to any $D \in \Delta_m$ a singular knot K_D with m double points. There are choices, but any two such singular knots differ by crossing changes, and the value of the invariant stays unchanged as I said before.

We can translate the relations before into chord diagrams. The one-term relation says that $w(v)$ is 0 on any diagram with an isolated chord. The four term relation you can just write down.

Call a functional on chord diagrams satisfying these two relations $1T$ and $4T$ a *weight system*. Call W_m the space of degree m weight systems.

Theorem 1. *$V_m/V_{m-1} \cong W_m$.*

We described a map from this quotient to W_m, and so you can check that if you have a type $m-1$ invariant, it will be zero as a weight system. Constructing a map that is an inverse is rather harder. The first to do this was Kontsevich, using the Kontsevich integral.

The open conjecture in this area is that every knot invariant can be approximated by finite type invariants. This is like the Taylor theorem. Equivalently, you could say that any two knots can be separated by a finite type invariant.

Let me say a few words about configuration space integrals. Linking number can be described as the composition of a map:

$$S^1 \times S^1 \hookrightarrow C_2(S^1 \sqcup S^1) \hookrightarrow C_2(\mathbb{R}^3) \stackrel{\varphi_{12}}{\to} S^2$$

where the final map is $x_1, x_2 \mapsto \frac{x_1 - x_2}{||x_1 - x_2||}$.

The linking number is the degree of this map. What Bott and Taubes did, they had some space, over the space of knots, a bundle, whose fiber is some compactified configuration space. So they integrated differential forms along the fiber of this bundle to get a closed 0-form, hence a knot invariant. The last thing that I should say is that their methods were used to construct all finite type invariants (Dylan Thurston's undergrad thesis, and [unintelligible]wrote a paper filling in the details) and later to construct classes in $H^*Emb(S^1, \mathbb{R}^n)$ where $n \geq 4$ (in real coefficients). Any two knots in $\mathbb{R}^4$ are isotopic, but it has higher nontrivial homology. The last thing, this construction using configuration space integrals is an alternative approach to the Kontsevich integral.

2. Geometry Physics seminar: A homotopy theoretic view of Bott-Taubes integrals and knot spaces

Of course, the integrals are a way of constructing the finite type invariants, so this is a generalization of a generalization of the finite type invariants. So let $K = Emb(S^1, \mathbb{R}^3)$ or $\mathbb{R}^d$, in which all the knots are isotopic but there are higher homotopy classes or homology groups. We also might want to consider embeddings $(\mathbb{R}, \mathbb{R}^d)$. where imagine fixed behavior outside of a compact set. We'll also be interested in the cohomology H^*K for these spaces. A higher degree cohomology class is like a generalization of a knot invariant.

So Bott and Taubes constructed knot invariants by integrating differential forms along the fiber of a bundle over the knot space $\begin{array}{c} E \\ \downarrow \\ K \end{array}$ with fiber F a compactified configuration space. The methods they used have been used to construct finite type or Vassiliev invariants, all finite type invariants, as well as classes in higher degree cohomology of spaces of knots in $\mathbb{R}^d$ where $d \geq 4$ with real coefficients. What I've been working on is using a Pontrjagn-Thom construction to construct classes with arbitrary coefficients, not just real coefficients. I've also proven a product formula, the classes I've been constructing, where the product is with respect to connect sum of knots. If time permits, maybe I'll talk about some later work, talking about tools coming from the Goodwillie embedding calculus, but this is roughly the outline of what I'll say.

So first, to talk about the Bott-Taubes integrals in some detail: the fiber that appears is a compactified configuration space. Well the configuration space $C_q(X) = \{(x_1, \ldots, x_q) \in X^q | x_i \neq x_j\}$, which is an open subset of X^q. If you have an inclusion of spaces or embedding of manifolds, this induces a map on configuration spaces. $C_q(X) \to C_q(Y)$. This is not a compact manifold. If we want to do integration, we need a compact version. There's something called the Axelrod-Singer or Fulton-MacPherson compactification. I'll say $C_q[M]$ is something that keeps track of directions of collision and relative rates of approach of points.

Now this is the dangerous part. So far, so good. One way of saying it is that this compactification is obtained by blowing up along all the possible diagonals. For example, the circle, we can visualize some of these spaces. $C_2(S^1)$, For the circle, you take out the diagonal, this is the open cylinder $(0, 1) \times \S^1$. The compactification is the closed cylinder.

A more subtle example would be three points on the circle, which looks like, well, a component looks like a hexagon cross a circle. Instead of just keeping track, well, when three points collide we have a whole edge of possibilities because of relative rates of approach.

The important point is that, so, the compactification is homotopy equivalent to the original configuration space and is also a smooth manifold with corners. This is defined for a compact manifold, and so we'll look at $\mathbb{R}^3 \subset S^3$. What Bott and Taubes do is consider a bundle over the knot space, and the total space $E_{q,t}$ is the pullback in the diagram:

$$\begin{tikzcd} E_{q,t} \arrow[r] \arrow[d] & C_{q+t}[\mathbb{R}^3] \arrow[d] \\ \mathcal{K} \times C_q[S^1] \arrow[r] & C_q[\mathbb{R}^3] \end{tikzcd}$$

A point in the pullback is a knot with $q+t$ points, the first q of which live in the image of K. We have this bundle. The fiber $F_{q,t}$ is a compact manifold, finite dimensional, with corners, and we can integrate—

[Ezra: what is that map? Is it differentiable? What does that mean?]

They integrate forms along the fiber of this bundle. They pull back—you have this configuration space, you pull back from S^2 along the map φ_{ij}, is the map to the unit vector difference. So in general, integration along the fiber is a map $\Omega^p E \to \Omega^{p-\dim\ F} B$. We'd immediately get a closed form by Stokes' theorem if this manifold was closed, but because the boundary is nonempty, $d\int_F = \int_F d \pm \int_{\delta F}$. So Bott and Taubes consider a particular 0-form gotten by, they integrate along the fiber $F_{4,0}$, they look at

$$\frac{1}{4}\int_{F_{4,0}} \theta_{13}\theta_{24} - \frac{1}{3}\int_{F_{3,1}} \theta_{14}\theta_{24}\theta_{34}$$

where θ_{ij} is the pullback of the top form of S^2 along φ_{ij} and the map from the pullback to $C_{q+t}[\mathbb{R}^3]$. This is a zero form, and they show that it's closed by showing that $\int_{\delta F} = 0$.

By the way, for those of you at the first talk, you can associate the integrals to chord diagrams, and so you can use graph cohomology to encode some of this information. But it essentially boils down to integration of differential forms. This particular invariant is a multiple of the first nontrivial Vassiliev invariant.

So now what I've been working on is doing this homotopy-theoretically.

Let me review the Pontrjagn-Thom construction briefly. If you have a bundle of finite dimensional compact manifolds, you can embed E into $\mathbb{R}^N$ and then use this to get an embedding of $E \hookrightarrow B\times\mathbb{R}^T$ over B. So ν is a tubular neighborhood of E which is diffeomorphic to the normal bundle of the embedding. So you quotient by the complement of the tubular neighborhood and get

$$\begin{tikzcd}
\mathbb{R}^N \times B \arrow[r] \arrow[dr] & \mathbb{R}^N \times B/(\mathbb{R}^N \times B - \nu) \arrow[r, "\cong"] & D(\nu)/S(\nu) =: E^\nu \\
& \Sigma^N B_+ \arrow[ur, "\tau"'] &
\end{tikzcd}$$

where you get a factorization through the one point comactification. Using the Thom isomorphism and suspension isomorphism, τ induces $H^*E \to H^{*-\dim\ F}B$.

In the case we're in, $E_{q,t}$ has corners from $F_{q,t}$, so we want to embed it, well, what we can do is embed $E_{q,t}$ into the product of $\mathcal{K}$ with $C_{q+t}[\mathbb{R}^3]$, and this can be embedded in a way that preserves the corner structure into Euclidean space with corners $\mathcal{K}\times\mathbb{R}^N\times[0,\infty)^L$. This is a "neat" embedding. In my thesis I used a categorical approach to manifolds with corners due to Laures.

[Ezra: if I remember there's a version where you can look at the Fulton-MacPherson for two and three, you can embed into a product of those]

Perhaps this is not something I was aware of. There are certain conditions where you need, say, a nice enough manifold with corners. There are things about codimensions of faces.

So in any case, there are certain conditions, codimension one faces, and so on. So from the Fulton-Macpherson, you can see that these things are satisfied. This gives a good notion of a normal bundle. Now we can collapse by the complement of a tubular neighborhood:

$$\mathcal{K} \times \mathbb{R}^N \times [0,\infty)^L \to E^{\nu_N}$$

but we'll get a one point compactification of $\mathbb{R}^T \times [0,\infty)^L$ is a cone $C^L S^N$, so one thing you can do, what I've written up, you can quotient by the boundary and have to quotient by certain boundary strata in the target space, which eventually gives you a map like

$$\Sigma^{L+N}\mathcal{K} \to E^{\nu_N}/\delta E^{\nu_N}$$

This gives you a map of spectra as $N \to \infty$, which induces a map in cohomology using a relative Thom isomorphism $H^*(E, \delta E) \to H^{*-\dim\ F}(\mathcal{K})$ which works for arbitrary coefficient rings or, if you wanted, cohomology theories with respect to which ν is orientable.

[So what are these?]

It comes from embedding the configuration space into Euclidean space, which is an open subset of Euclidean space, so...

[Are there nontrivial examples?]

I would not think that you should crush the boundary all the way to a point. I'm trying to do some gluings of configuration spaces. There's a paper on three-manifold invariants by cut and paste technology. This suggests that things like this could work.

So for example, their examples are from the sphere pulled back. Working in E relative its boundary, I couldn't get, well, the dimensions wouldn't be quite right. A more refined gluing would be a better way to do this.

To motivate, what I do have is a product formula. If I knew how to evaluate one class I could evaluate other classes. The motivation is Budney and Fred Cohen. Budney found a little 2-cubes action on $\mathcal{K} = Emb(\mathbb{R}, \mathbb{R}^3)$. (Of course, Bott-Taubes works for long knots and any $\mathbb{R}^d$) He shows that $\mathcal{K}$ is the free 2-cubes object on the space of prime knots, and this result combined with results from Fred Cohen's thesis allow them to say that $H_*\mathcal{K}$ is, roughly speaking, generated by $H_*\mathcal{P}$, the homology of the space of prime knots. All the homology classes in the space of prime knots, take products and brackets, and this will give you all homology classes in the space of knots (A knot is prime if it can't be written as a nontrivial connect-sum).

[Is that a map of E_2-ring spectra?]

It's a map of ring spectra, that's exactly where I'm going. This motivates the goal, if you have a class β in $H^*(E, \delta E)$, and have two knot homology classes $a_1, a_2 \in H_*\mathcal{K}$ and ξ_* a homology operation $H_*\mathcal{K} \otimes H_*\mathcal{K} \to H_*\mathcal{K}$; given this information you can use duality, integrate along the fiber, and you want to evaluate, looking, at, say, $\langle \tau^*\beta, \xi_*(a_1 \otimes a_2)\rangle$, and you can write this as $\langle \xi^*\tau^*\beta, a_1 \otimes a_2$, and the goal would be to calculate evaluations on "Bott-Taubes classes" on all knots in terms of evaluations on prime knots.

[Discussion of the two-cubes action.] It makes for a good picture in a slide talk, but this is my low-budget version.

The τ from the suspension spectrum $\Sigma^\infty\mathcal{K}$ of $\mathcal{K}$ to the Thom spectrum of the total space modulo its boundary $E^\nu/\delta E^\nu$ is a map of ring spectra. It respects the multiplication, so we can define a multiplication on $\coprod E_{q,t}$ compatible with space level connect sum $\mu : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$. This induces multiplications on the configuration spaces themselves, which allows computing the product formula, which says basically that $\langle \tau^*\beta, \mu_*(a_1, a_2)\rangle$ is $\sum\langle \tau^*\eta_i, a_1\rangle\langle \tau^*\zeta_i, a_2\rangle$ where η_i and ζ_i can be computed in terms of $H^*(C_{q+t}[\mathbb{R}^3])$. Maybe just to finish answering John's question, I wasn't able to lift the two cubes action, the configurations are ordered, so the multiplication on the disjoint union of the configurations, it's not commutative in homology.

Thanks for your attention.

[More questions?]

[Could you explain more about the Goodwillie calculus? Is this construction related?]

[That might be a good one-on-one question.]

The Goodwillie calculus gives rise to a cosimplicial model where configurations appear as the levels in the space that model the space of knots (at least for $\mathbb{R}^4$ or higher) and this doesn't use integration along the fiber, but there is, [unintelligible]'s thesis showed that for $\mathbb{R}^3$, Vassiliev invariants factor through the Goodwillie tower (where we don't know we have convergence).