TOPOLOGY SEMINAR
ROSONA ELDRED
A TALE OF TWO TOWERS

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So I'm kicking off your seminar. [Are you a Dickens fan?] I never got through it, they killed the character I liked. So, I will start by saying what I'm going to prove, because I'm told that's a good way to start. Then I'll give you a big picture and consequences and then if there's time, I'll give a proof. So I really hope I can assume people know what a homotopy limit is.

I talked to Tom Goodwillie about what I was working on. He made a conjecture and I proved it.

Theorem 1. *Let F be a* homotopy functor *from spaces to spaces. What do I need by this? If $X \sim Y$ then $F(X) \sim F(Y)$. So I'm using homotopy equivalence. Then the following are equivalent:*

$$\operatorname{holim}_n T_n^{k+1} F(X) \sim \operatorname{holim}_n \operatorname{Tot}_n F(sk_k \Delta * X)$$

The category Δ is the category of finite ordered sets and monotone maps, and Δ is the functor to spaces which takes $[n] \to \Delta^n$. So $sk_k\Delta(n)$ is, well, Δ^1 is the interval, and $sk_0\Delta^n$ is the points. $*$ is join, you take the spaces and make the lines between everything.

Let me say this a diffeernt way, you can say that this is $sk_k\Delta \otimes_X CX$. So $\{0,1\} \otimes_X CX$ is the pushout of CX and CX over X which is the suspension of X. This is related to spectra. So one side is reasonably easy to define. What about Tot? If you know about simplicial sets or spaces, you have a way to move them back into spaces. If you replace your cosimplicial space with something fibrant and take Tot, that's the same as taking $\operatorname{holim}_{CS} X$. So for me $Tot(X)$ is $\operatorname{holim}_\Delta X$. Then $Tot_n(X)$ is the homotopy limit over the truncated subcategory where $|k| \leq n$.

The left hand side is harder to deal with. Maybe I'll tell you the role instead of what this is. So T_n^{k+1} is the calculus of functors stuff.

Back in normal calculus land, we all remember the nth Taylor polynomial $P_n f$ as $\displaystyle\sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} x^i$. This is analagous, although only special things will have all of the P_nF come together to make F. The analogue to $P_nF = \operatorname{hocolim}_k T_n^k F(x)$. So for instance $T_n^2 = T_n(T_nF(X))$. A reasonable story is the following. I will think of $x+h$ physically as the cone on x with cone point h. So think of $x+h-h$ as SX. Then $T_1F(X)$ is the homotopy pullback of F applied to these cones mapping into SX.

The picture I was working with was, I was looking at cosimplicial spaces. I should do this in the other order. Let me give you the big picture. Maybe 12 years ago, Waldhausen, then and earlier he indicated how one could obtain (derived) de Rham cohomology of a rational map using a process related to Goodwillie's Taylor Tower. The P_n I mentioned earlier assemble into a tower $\cdots \to P_nF \to \cdots \to P_0F$. The homotopy limit is $P_\infty F(X)$. Waldhausen said, "if I work rationally (and classically), then the Taylor tower P_∞ of the identity on commutative A-algebras or forgetful functor from commutative A-algebras to A-modules will give the de Rham cohomology over A functor.

The property of being degree n has to do with taking a pushout cube to a pullback cube. If you follow it by something that preserves pullbacks, then the Taylor tower commutes with the functor preserving pullbacks.

This is not actually close to the identity, because it doesn't converge. These algebras are commutative rings with a map $A \to B$, unbased. There is something else problematic if you know about Goodwillie calculus.

Let's look at the other side of the picture: Rezk (unpublished) works classically (discrete rings) and rationally and considers this complex $sk_1\Delta \otimes_A B$. He says this should be de Rham. Joint work with others proves that these two notions are the same even not rationally and for E_∞-algebras.

[Can you parse notation?]

$sk_0\Delta \otimes_A B$ is the Adams resolution, the Amitsun complex, or the B-nilpotent completion of A. I can tensor spectra over sets. At the first level, $sk_0\Delta^1 \otimes_A B$ is the pushout of $B \leftarrow A \to B$.

[Is there a reason not to use the normal definition?] Well, de Rham cohomology works well for rational and commutative things. The problem is that it's not well defined for E_∞ algebras. I'm not studying de Rham cohomology, I'm noticing that this cosimplicial thing over here is related to that Goodwillie calculus thing over there and maybe I can do something more than this.

This was only supposed to be motivation. Randy thought I shouldn't even include this, but I thought it was good. Let me give some consequences. So, corollary, this is a space-level version of some of the things in this joint paper for spectra.

Corollary 1. *Let F be a homotopy functor. For any k, $Tr\ P_jF(X) \cong P_jF(X)$, so*

$$P_jF(X) \cong \operatorname{holim} T_n^{k+1} P_jF(X) \cong \operatorname{holim} Tot_n P_jF(sk_k\Delta * X)$$

One more thing. So Randy said I should make this a theorem, which is why I forgot to move it up. This is a strange translation.

Theorem 2.

$$T_n^k F(X) \cong \operatorname{holim} F(\underbrace{sk_0\Delta * \cdots * sk_0\Delta * X}_{k+1})$$

Corollary 2. *If F is ρ-analytic, if you apply F to a ρ-connected space then $F(X) \cong P_\infty F(X)$, that's a working definition, then I can recover the Taylor tower, well, $P_\infty F(X) \sim \operatorname{holim}_n T_n^{\rho+1} F(X)$.*

I'm surprised, I have enough time to talk about what I did. In the process of this, I discovered a model category term that I had not used before, which is *cofinality.* It's not clear what you need when you say this with no modifiers. I'll say homotopy left cofinality. If you're talking about limits you'll talk about right cofinality. Other people say homotopy initial. I'm just using what is found in Hirschhorn. This is the giant hammer that I'm going to beat this problem to death with so I need to be careful setting this up. I'll use a different definition since this is a really nice category. If you want a section, this is supposed to be an overview.

Definition 1. *Let G be a functor from $\Delta \to \mathcal{D}$ (for us this will be spaces). The functor is homotpy left cofinal if, normally I would say $B(G \downarrow \alpha) \cong *$ for all $\alpha \in D$. An equivalent statement is if for α in the objects of $\mathcal{D}$, the simplicial set $n \mapsto Mor_{\mathcal{D}}(G(n), \alpha)$ is contractible.*

Why do I care and what does it buy me?

Theorem 3. *(classical, in Hirschhorn)*
Let M be a simplicial model category, and let $\mathscr{C}$ and $\mathcal{D}$ be small categories, with Z a functor $\mathscr{C} \to \mathcal{D}$. If Z is homotopy left cofinal, then for every $\mathcal{D}$-diagram F in M there is a, $\operatorname{holim}_{\mathcal{D}} F \to \operatorname{holim}_{\mathscr{C}} F^* Z$.

Let me translate my problem over there into one that involves cofinality. Let me define $\mathscr{X}_k(p)$ as $sk_k\Delta^p$ and $\mathscr{Y}_k(p) = \underbrace{sk_0\Delta^p * \cdots * sk_0\Delta^p}_{k+1}$. Then I can rewrite the right hand side inside the holim in my original theorem as $\operatorname{holim}_{\Delta_{\leq n}} F(\mathscr{X}_k * X)$.

If I have a space X then I can define $F_X(\quad)$ to be $F(\quad *X)$. Then this holim is $\operatorname{holim} F_X(\mathscr{X}_k)$ and the other side is $\operatorname{holim} F_X(\mathscr{Y}_k)$. This is the point we have to pull this theorem in. Now I've gotten this into a form I can beat to death with cofinality. They will both be equivalent to a third homotopy limit so equivalent to each other.

Lemma 1. *$\mathscr{X}_k$ and $\mathscr{Y}_k$ are homotopy left cofinal (almost) for $k \geq 1$.*

I started by taking apart cosimplicial objects, and looking at sk_k versus Tot. On the diagonal line, and above you have $F(*)$, and below it's $F(\emptyset)$. What's wrong with this that there's an almost? I end up showing that these cleverly named simplicial objects $Map(\mathscr{X}_k(p), \alpha)$ where α is a k-dimensional simplicial complex, if you show this is fibrant, to calculate its homotopy groups you can look at simplicial maps Δ^p in with all faces degenerate. When the skeleta get too big which is 2, you map things in that start recovering α, but only for a finite level. At each stage if I can throw away a finite amount, I'm good.

Lemma 2. *$\Delta_{\geq k}$ is still homotopy left cofinal.*

I think that's me being out of time and I actually gave a sketch of the proof.

[How did this come up?]

There's another functor that lets me blow up my Tots in terms of cubes. I made the observation that when you're dealing wih sk_0, then I get exactly T_n. That's weird, I said, okay, and Randy said, okay, maybe you can find some maps between these layers, try to find things that behave kind of like differentials. I got lift maps, and ended up showing something interesting, that $Tot_1 F\ sk_0 \cong T_1 F(X)$ and I showed that all of these spaces along this diagonal are [unintelligible], so I can recover P_1 as the homotopy colimit along the Tots. I can also, well, for T_1 I get slope one, for T_2 I get slope two. For T_3 I start seeing slope 3 things. This result is about doing the horizontal. I was talking to Tom, maybe they're comparable, like equal object wise at a finite level.

So I was trying to map between the towers, and on the one-skeleton level there was a nice map that composed to the identity up to homotopy. Tom said you might have an equivalence of towers. Then he said that he always thought the horizontal towers were easier to study.

This is also supposed to play into unbased Goodwillie calculus and Randy trying to classify all n-functors.