GEOMETRY/PHYSICS SEMINAR
YONG-GEUN OH, WISCONSIN

GABRIEL C. DRUMMOND-COLE

1. Pretalk: Counting embedded curves in Calabi-Yau threefolds and Gopakumar-Vafa invariants

The theeorem I'll talk about in my main talk is to prove a rigidity for [unintelligible]-holomorphic curves for generic choice of almost-complex structure.

Let's say we have six dimensional (X, ω) with $c_1(\omega) = 0$, so symplectically Calabi Yau 3-folds. Let's look at $\mathcal{J}_\omega$, the space of almost complex structures with J compatible to ω, that is $g_J = \omega(\cdot, J\cdot)$ is symmetric positive definite. We're interseted in looking at maps $u : \Sigma \to (M, J)$ with $\bar{\partial}_J u = 0$ where (Σ, j) is a closed Rieman surface of genus g.

What's special about Calabi-Yau 3-folds? Well, $\tilde{\mathscr{M}}_J(\Sigma : \beta)$, the set of such maps with $[u] = \beta \in H_2(M)$, and then when you mod out by $Aut(\Sigma)$, the quotient $M_J(\Sigma, \beta)$ has virtual dimension 0. Because of this, if this space is a compact manifold, there are only a finite number of curves in that class. The basic invariant we want to define is this number.

The main difficulty is, how do we handle the multiple covers? This means that we have $\Sigma \to C \to M$, where u is the composition of $\phi : \Sigma \to C$ and $\tilde{u} : C_g \to M$, where ϕ is a branched covering from a surface of genus $g + h$ to a surface of genus g. Here $\tilde{u}$ is somewhere injective, so that there is an immersed point somewhere with preimage one point.

This is what I hope to talk about in the main talk. The problem here is to prove for generic choice of J, the kernel of the linearized operator $(D\bar{\partial}_J(u))^\perp$ vanishes.

A somewhere injective J-holomorphic curve will be an embedding so it has a normal bundle, then this map goes from $\Omega^0(u^*N_C) \to \Omega^{0,1}(u^*N_C)$. This is called rigidity. This rigidity generically is equivalent to some transversality but the rigidity for multiple cover is independent of the transversality property.

Let me introduce some contact geometry. A contact manifold is (M, ξ) where ξ, the contact structure, is a distribution of codimension one, so there exists a 1-form λ so that $\ker \lambda = \xi$, and if you take $\lambda \wedge (d\lambda)^n$, this is nowhere vanishing on M, so it defines a volume form.

There is more than one such λ, so let me define $Cont(\xi)$ which is the space of λ so that $\ker \lambda = \xi$. You can see that this is isomorphic to $C^\infty(M, \mathbb{R}_+)$. So on the left we have the space of almost complex structures and on the right the space of contact structures.

Let's say you are given (M, ξ) with λ a contact form, then this contact form defines a vector field on M, the Reeb vector field X_λ, characterized by the properties that contraction of d_λ gives 0 and by λ gives 1.

The basic problem in contact geometry is the study of closed orbits of $\dot{x} = X_\lambda(x)$.

Now as in the study of pseudoholomorphic curves, you can set the following up. In this case, the period of the Reeb orbit is not given, so the period is a parameter.

Definition 1. *A closed orbit of $x = X_\lambda(x)$ is a pair (T, z) where T is a positive real, and z is a loop with period T, $z \in \mathcal{L}_T(M)$.*

This is the set of maps $\mathbb{R}/T\mathbb{Z} \to M$, with $\dot{z} = X_\lambda(z)$ Since the period could appear with any numbers, you should regard $\mathcal{L}$ as the fiber bundle of $\mathcal{L}_T$ over $\mathbb{R}_+$. Then you can define a bundle V, the union over $(T, z) \in \mathcal{L}$ of sections of z^*TM. This is a bundle over $\mathcal{L}$ whose fiber is sections of the vector bundle along z.

I can define a section of this bundle υ where $\upsilon(T, \gamma) = \gamma - X_\lambda(\gamma)$. Then the inverse image $\upsilon^{-1}(u)$ is in correspondence with closed Reeb orbits.

One can also look at N_z which is always isomorphic to $z^*(\xi)$ if z is a simple curve. Of course, in general, yeah. The basic question is the following: to prove nondegeneracy, the kernel of the linearization, the cokernel, prove generic nondegeneracy of closed Reeb orbits (T, z). In other words, prove there exists some dense subset of contact forms $Cont^{reg}(\xi)$ so that $D\upsilon(T, z)$ has trivial cokernel, which is the same as saying trivial kernel.

There is a geometric description of this nondegeneracy. Suppose you are given the closed Reeb orbit. Look at the normal sections, and that is tangential to the contact structure. Al the nearby points go around. Look at $\gamma : [0, T] \to M$ a closed Reeb orbit and consider the Poincaré return map $P_\gamma : D \to D$. The basic proposition is that this is $D\upsilon(T, z)$ has trivial cokernels if and only if, you look at the derivative, $DP_\gamma(0)$ (this is a linear map frome $T_{\gamma(0)}D$ to itself) has no eigenvalue 1. That's a geometric description of this nondegeneracy.

Generic nondegeneracy means that for a given contact form we want to have this for all closed Reeb orbits. There may be some accumulation of these, and then somehow the proof of the generic nondegeneracy sholud go as follows. First you prove the nondegeneracy, and then you want to perturb the Poincaré return map to make the eigenvalues nonrational. But there could be an accumulation, then you cannot make this perturbation, where you have to choose smaller and smaller neighborhoods. You may not be able to localize.

How do you achieve the [solution to] problem? The scheme of the proof goes as follows. In step one, we first want to prove this generic nondegeneracy for simple Reeb orbits. This is kind of a standard argument. It's like the proof of [unintelligible]for pseudoholomorphic curves. This gives rise to a first try of a dense subset $Cont^{simp}(\xi)$ of $Cont(\xi)$. We know all the simple Reeb orbits will be nondegenerate.

So the next step is, you do this local perturbation of the normal bundle, I guess I mean λ' of the contact form, I mean, near each closed Reeb orbit, near the locus C of each simple Reeb orbit, so that $\lambda'|_C$, well, z is still a λ'-Reeb orbit, and $DP_\gamma(\lambda')$ has no rational eigenvalues. This is an easy perturbation argument. We now want to glue these perturbations. The next step is what happens in the neighborhood? We'll do a perturbation in a tubular neighborhood of each loculs. So here comes the miracle. The space $Cont^{simp}(\xi)$, you want to have a manifold of parameter space. The way to get out of this problem is the following. Instead of considering all possible [unintelligible]simultaneously, you do it step by step.

One important piece of data is that $T = \int z^*\lambda$. The basic compactness of these ODEs, if you keep the bound on periods and the space of Reeb orbits [unintelligible], then [missed something.]

Lemma 1. *For any $T > 0$ the space of Reeb orbits with period at most λ is compact.*

The modification this suggests is the following. We study simple generic nondegeneracy for integral periods $\leq T$.

Previously, $Cont^{simp}(\xi) = \bigcap Cont^{simp}_{T\leq N}(\xi)$.

Lemma 2. *$Cont^{simp}_{T<N}(\xi)$ is open and dense. You can use this as a parameter space to do the next things.*

Here is a modified step two: consider the fixed $T \leq N$. Consider $Reeb_{T\leq N}(\lambda)$ for $\lambda \in Cont^{simp}_{\leq N}(\xi)$. Then $\#(Reeb_{T\leq n}(\lambda) < \infty$. Then we carry out step two for each Reeb orbit, simple Reeb orbit, in tubular neighborhoods that are pairwise disjoint.

So by gluing these perturbations, these give rise to contact structures λ' which we hope have only irrational eigenvalues for all simple Reeb orbits of period $T \leq N$.

This does not solve the problem completely. We haven't talked about multiorbits and their nondegeneracy. I made the perturbation near the locus of this simple Reeb orbit. Then the multiple Reeb orbits will be perturbed into a simple Reeb orbit looking like a coil. This simplest Reeb orbit may have rational eigenvalues. There are some simple Reeb orbits whose periods are small.

That's the problem. The subtlety is there is a new simple Reeb orbit which is close to a k multiple λ-simple Reeb orbit whose period has $kT \leq N$.

If you want to do a perturbation around the new one, this kind of thing may keep happening.

The problem is, you have to handle degeneration of simple Reeb orbits to a multiple covering.

That can be handled by introducing another parameter. Here I used period, $T = \int z^*\lambda$. We can use another parameter $L = \inf \frac{d(z(t),z(t'))}{d(t,t')}$. If this is positive, thene z is simple. So if we can bound this away from zero, then we will not have a problem, so now lot's do a second modification. Now you consider Reeb orbits with $T \leq N_1$ and $L \geq \frac{1}{N_2}$. Then you do this perturbation, $Cont^{L\geq\frac{1}{N_2}}_{T\leq N_1}(\xi)$, [unintelligible]is nondegenerate. For any λ' here and any Reeb orbit (λ'-Reeb orbit z) is nondegenerate. I jumped one step. Now you can do a perturbation. You don't have to worry about the degeneration because it won't occur with L bounded away from 0.

2. Seminar: Counting embedded curves in Calabi-Yau threefolds and Gopakumar-Vafa invariants

Thank you for the invitation. Okay, so I don't know much about the physics behind these invariants, but I will talk about some part of Gromov-Witten theory. Let's let X be symplectically Calabi-Yau, so we have (X,ω) with X of dimension 6, closed, compact, with $c_1(\omega) = 0$. Then the Gromov-Witten theory defines some invariants. Let β be a homology class in $H_2(X)$, and

I'll ignore the 0 case, $\beta \neq 0$. Then choose $J \in \mathscr{J}_\omega$, a compatible almost-complex structure. Then we look at $\tilde{M}_g(J,\beta)$, the set of maps u from a Riemann surface Σ_g of genus g into X so that $\bar{\partial}_J u = 0$ and $[u] = \beta$. We'll let $\mathscr{M}_g(J,B) = \tilde{\mathscr{M}}_g(J,B)/Aut\ \Sigma_g$. The virtual dimension of $\mathscr{M}_g(J,B)$ is always 0 for any genus g and $\beta \in H_2(X)$ (although it may not be a manifold). Then you compactify to get $\bar{\mathscr{M}}_g(J,B)$

Let $N_\beta^g(X)$ denote the virtual integration:

$$\int_{[\bar{\mathscr{M}}_g(J,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

which is not an integer because of automorphisms. Then you define formally

$$\mathscr{F}_g(q) = \sum_\beta N_\beta^g(X) q^\beta$$

where q^x is something of the form $e^{-i\omega(\beta)}$ but that is how a physicist would write it, it's not well-defined I would say a formal power series

$$\sum N_\beta^g(X) T^{\omega(\beta)}$$

and we have a reduced version $\tilde{\mathscr{F}}_g(q)$ where we sum over $\beta \neq 0$.

So then we can write

$$\mathscr{F}(\lambda) = \sum_{g \geq 0} \lambda^{2g-2} \mathscr{F}_g(q)$$

and

$$\tilde{\mathscr{F}}(\lambda) = \sum_{g \geq 0} \lambda^{2g-2} \tilde{\mathscr{F}}_g(q)$$

That's the story. Then Gopakumar-Vafa, some years ago, they computed this full generating series using physical theories, some M-theories, they predicted, they wrote down this series in a form that is a closed form. The M-theory prediction is that

(1)
$$\tilde{\mathscr{F}}(\lambda) = \sum_{\beta \neq 0} \sum_{g \geq 0} n_\beta^g(X) \sum_{k>0} \frac{1}{k} (2 \sinh \frac{k\lambda}{2})^{2g-2} q^{k\beta}$$
You can determine the n_β^g and N_β^g from one another providing things are rigid. There is a sort of inversion of the one to the other provided first that simple curves are embedded and all these together with their multiple covers are rigid. We can call this "trivial kernel," which I'll explain more. In this case you can determine these quite explicitly in terms of one another.
(2) $n_\beta^g(X)$ are integers, even though $N_\beta^g(X)$ are rational, by BPS counting.
(3) For fixed β, $n_\beta^g(X) = 0$ for $g \gg 0$.

There is then some sort of folklore impression that this invariant should be related to the counting of embedded curves. I don't know if it's their conjecture or others', but the conjecture is that n_g^β "should be" the number of embedded curves of genus g in class β. Under this rigidity, Bryan-Pandaharipande gave indication that this conjecture should be true, provided that the number of embedded curves of genus g is well-defined, meaning that, first of all, the number of J-holomorphic embedded curves is finite in β, and indpendent of the choice of J. Unfortunately, it's not clear if the second holds, I think it does not. The first one will be proved for generic choice of g.

The main theorem is that there exists a dense subset $\mathcal{J}_\omega^{rig} \subset \mathcal{J}_\omega$ such that $\tilde{n}_\beta^g(J)$, the number of J-holomorphic curves embedded in class β with genus g is finite. The open problem in the moment, we expect, there is some kind of criterion which defines the complement of this set in $\mathcal{J}_\omega$. If you cross those walls, n_g^β will vary. However, N_g^β can be written in terms of n_g^β plus some local contributions, and [unintelligible].

[What does N_g^β count?] Holomorphic maps, not necessarily embeddings.

Theorem 1. *(Oh, Zhu)*

(1) *Let X, with dimension at least six, have $c_1(\omega) = 0$. Then there is a dense subset $\mathcal{J}_\omega^{emb} \subset \mathcal{J}_\omega$ where for $J \in \mathcal{J}_\omega^{emb}$, any somewhere injective J-holomorphic curve is embedded.*
(2) *In general the structure of stable maps in quite complicated. In dimension six, with $c_1(\omega) = 0$, then any connected stable map has a non-constant irreducible component have the same locus. [sic]*

This means there is an embedded J-holomorphic curve whose image locus is C inside X and any holomorphic map will be a branched cover over C, so

$$u : \Sigma_{g+h} \xrightarrow{\phi} C_g \xrightarrow{\tilde{u}} X$$

where ϕ is a branched cover and $\tilde{u}$ is an embedding.

We are trying to prove the embedded curves are finite, and I need to add one more statement in the main theorem, that all those embedded curves are superrigid in the language of Bryan-Pandaharipande, not only the curve is rigid, but its multiple covers are as well.

For the rest of the talk I will try to explain this finiteness of embedded curves and superrigidity. The structure of the normal bundles will be more complicated than in the pretalk.

Let's look at degenerations of embedded curves. Let's say u is in class β, and say that $u : \Sigma_g \to X$, and u is an embeddding. We want to study the compactification as embedded curves. There are three degenerations.

(1) The first one is that the derivative blows up. This is "bubbling."
(2) The derivative goes to 0, this sort of could happen and this leads to self-intersection. This can be ruled out by dimension counting, but more seriously it could converge to a multiple cover.
(3) Since we can change the complex structure in the domain, the branched cover ϕ could degenerate.

We needed to prove finiteness and superrigidity simultaneously because they are connected to each other. The scheme of the proof is the following:

(1) Prove rigidity for simple curves.
(2) Handle their multiple covers.

In this perturbation theory, we want to apply [unintelligible]repeatedly, so we want to introduce a Banach structure. Instead of considering all these curves simultaneously, we are going to break this up by imposing an energy bound and other bounds to control the degeneration. First, by the way, rigidity for simple curves, there are two notions, look at the linearization. $D\bar{\partial}_J(u) : \Omega^0(u^*TX) \to \Omega^{(0,1)}(u^*TX)$, that's the linearization, we want the normal of that $(D\bar{\partial}_J(u))^\perp : \Omega^0(u^*N_C) \to \Omega^{(0,1)}(u^*N_C)$.

Definition 2. *We call u rigid if the krenel of this normal is 0, which in our context is equivalent for embedded curves to the cokernel vanishing. For multiple curves this is no longer the case.*

These parameters control the degeneration:

(1) energy, $K = \omega(\beta)$, the symplectic area, we impose $\omega(u) \le K$.
(2) the genus bound, genus at most G, and
(3) third, this will control the derivative blowing up or going to zero, this is the well-known dilatation constant, $\frac{d(u\circ\psi(z), u\circ\psi(z'))}{d(z,z')}$, this constant is greater than $\frac{1}{K}$ and less than K.
(4) Another parameter controls the variation of branched coverings, $M_{g+h}(C, d[C])$. This space is not compact, so you write down this thing as the union of subsets $\bigcup \mathcal{U}_L(C, d[C])$ where $\bar{\mathcal{U}}_L$ is compact. Then we want $((\Sigma, J), \phi) \in \bar{\mathcal{U}}_L$.

We look at all this, and prove rigidity with respect to G, K. Once you fix G and K you can talk about L. This is a dense open subset of $\mathcal{J}_\omega$.

What do I want to do, let me write down, denote by $\mathscr{M}^{emb}_{G,K,L}(J)$ this one is the set of embedded curves, J-holomorphic, with all these bounds. Now you can apply these generic embedded properties and show that this one is compact.

Now we can say $\mathcal{J}^{rig}_{G,K,L}$ is the set of almost complex structures such that for J in this set, any element $\tilde{u}$ in $\mathscr{M}^{emb}_{G,K,L}(J)$ is rigid for all multiple covers ϕ as long as $\omega(\tilde{u}\circ\phi) \le K$ and the genus of Σ is less than G, and the domain of $(\Sigma, \phi) \in \bar{\mathcal{U}}_L$. The embedded curves are not just rigid, but all of their multiple covers as well.

Let's see that this is nonempty. For embedded curves it follows from the standard transversality proof. For the standard curves, transversality implies rigidity. But we needed to handle the kernels of multiple covers. This one is quite similar to the one I talked about in the pretalk. The idea is similar here. How do you choose a perturbation? By the way, one of the consequences, I said embedded curves satisfy transversalities. Because of the bounds the space $\mathscr{M}^{emb}_{G,K,L}(J)$ is finite. Once you prove the transversalities, it's a 0-dimensional manifolds, so this immediately gives that $M^{emb}_{G,K,L}(J)$ is a zero dimensional manifold. There can be no degeneration to multiple covers, once we prove the transversality.

In the very beginning, I didn't mention that the embedded curves cannot intersect, so you have a collection of disjoint embedded curves. The perturbation is local. Take a tubular neighborhood of each of these, and make a perturbation of the almost complex structure, and the basic lemma is the following. Say C is the image locus, and it's J-holomorphic. For any neighborhood of J in $\mathcal{J}_\omega$ there exists a J' which is in $\mathcal{J}^{rig}_{G,K,L}$. This will be an open neighborhood of J so that the resulting curve is J'-holomorphic. Second, this normal bundle $(D\bar{\partial}_{J'}(u))^\perp$, the normal linearization, has trivial kernel as long as $u = \tilde{u}\circ\phi$ and u satisfies the three conditions from above, the G, K, L condition.

The normal to the Riemann surface is a holomorphic vector bundle with some complex structures i_{N_C}.

We deform the holomorphic vector bundle by the ambient almost complex structure J'. Then you study the curvature of the associated canonical Chern connection of this Hermitian normal bundle. This curvature enters, well, the proposition is, there is a criterion, this rigidity, the vanishing of these kernels $(D\bar{\partial}_{J'}(\tilde{u} \circ \phi))^{\perp}$ can be described in terms of a nonvanishing of a pairing: $\langle c, R^{(0,1)}_{(j,\phi,J,C)}(K)\rangle_2 \neq 0$.

You look at this for c in the cokernel of the normal map and K in the kernel of it. So there is some stratification Fredholm operators. This is connected to, regard this as a map depending on J', and so I can write this as a map of J' and ϕ. The point is that, looking at the diffeomorphism of domain, the connection, if you pull back the Chern connection by diffeomorphisms, these change the curvature properties completely. That somehow controls a lot, allowed me to choose J' so that rigidity holds. This involves differential geometry calculations which are not very common in symplectic geometry.

[Openness?] It's a consquence of compactness. Rigidity is an open phenomenon.

[Why can ϕ be nonholomorphic?] This is the main point. One of the difficulties in attacking this problem is the multiple covers, which give rise to symmetries on the variation, and because of the symmetries the transversality argument fails. The point is, you are looking at $(\Sigma, j) \to (X, J)$ factoring through C, which accepts a normal bundle. A normal part of linearization is $u : \Sigma \to X$. Then $(D\bar{\partial})^{\perp}$ is, we will call it, $\Psi(j, (\phi, J)C)$ and [unintelligible]$Diff(\Sigma) \times \mathcal{J}_W \to Fred(\Omega^0(\phi^* N_c), \Omega^{0,1}(\phi^* N_C)$ I'll define this one as $(\psi, J) \mapsto \Psi(j, \phi, \psi : J, C)$ this [what?] can be defined for any smooth u. This normal linearization can be defined for holomorphic maps and any map which has this composition properties. The point is, you can localize the perturbation.