MOHAMMED ABOUZAID
MATH PHYSICS PRETALK AND SEMINAR

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1. Pretalk: Symplectic Khovanov something

I can say some words about Khovanov homology. I can say some things about quilts and functors between Fukaya categories. I can say some words about Fukaya categories of cotangent bundles and their relation to the topology of Q. I'm not sure what else. The last thing that I should be able to say some things about is Springer varieties, springer fibers.

Okay, so Khovanov homology is an invariant of knots. These are embeddings of S^1 into S^3. The origin of knot theory was 19th century physics. In the thirties, Seifert and Alexander mostly started studying knot theory from the point of view of pure mathematics. They tried to use algebraic topology of the complement. So you may have heard about the Alexander polynomial. Really it's an Euler characteristic of a cover, a $\mathbb{Z}$-fold cover of $S^3 - S^1$. Regardless of the embedding, H_1 of this space is $\mathbb{Z}$. I can't draw knots, so π_1 is rich and complicated, but H_1 is generated by the meridian, and there's a canonical map, you look at how the deck transformations act on topology.

This is how knot theory was studied until the 80s. Kids studied it by drawing diagrams and adults studied it by studying algebraic topology.

Then Jones came along and said that you can construct a polynomial which is an invariant of knots, and you compute this by drawing a knot diagram, you give yourself the data, start with the polynomial for the unknot, 1, and then you go through the diagram, and resolve all of the crossings in all possible ways. For example, in this picture [trefoil] there are three crossings, so there are eight ways to resolve this. Locally the picture always looks like this, and you resolve by smoothing to remove the crossings. There are no crossings left and you are left with one big circle or many circles, an unlink. Now once you get to the bottom of the picture. It should be 1 for the unknot, and you have to figure out some way of getting the answer when you use a resolution.

The formula, which I will steal because I am not a knot theorist, says that

$$t^{-\frac{1}{2}}p(K_\infty) + t^{\frac{3}{2}}p(K_0) + t^{-1}p(K) = 0$$

This is the relation for an oriented knot. You produce something eventually that doesn't depend on the orientation. The fantastic thing is that you can plug it into a computer. The upsetting thing is that it doesn't have any connection to topology. What is this, where are we going, these were the questions we asked. This was in the 80s.

The knot theory field sort of split here into these quantum invariants and the topological section.

Khovanov categorified the Jones polynomial. So again, what do I mean? Numbers are like 0-categories. Instead of a number, we should associate a homology group. The Jones polynomial should be an Euler characteristic. He followed the same strategy. You start with the answer for the unknot. You'll notice some small differences already.

Let's take V to be $H^*(S^2)$ for the unknot. You could call this $\mathbb{C}[\epsilon]/\epsilon^2$ with degree of ϵ equal to 2. So for the n-unlink, you want to take $V^{\otimes n}$ which is the cohomology of $(S^2)^n$. This is our invariant for unknots or unlinks.

How do we produce an invariant for every knot? We produce a "hypercube" of graded Abelian groups from every knot.

Here we had 8 different resolutions. One of them, if you do all the crossings the same way you'll get the same answer. So for one of them you'll get the 3-unlink. The guy obtained from the opposite resolution is the 2-unlink. The adjacencies are obtained by doing one thing differently. So we have $V^{\otimes 3}$, $V^{\otimes 2}$, and so on. Our invariant will be a chain complex and I just need to give you differentials, but I won't.

Every time you do a different resolution, you put a differential on the edge of the cube. You should imagine that this cube is something like the E_1 page of a spectral sequence. You can take a trivial differential. You can take a trivial differential to get down to $H^*(S^2)$, and that this is only on the edges says that the sequence collapses at E_2, in terms of spectral sequences Now I will move up one higher level. This is often a consequence of formality of A_∞ structures. A module over an A_∞ structure, if both are formal, I would get something like this.

This is a brief overview. Before we could have thought that this polynomial was a coincidence. Now that you have a homology group you want to know if this is related to topology of knots. That's the ond of my overview. I will say some words about symplectic Khovanov homology. To show this is invariant, you have to go through the Reidemeister moves and check that the complex is up to chain equivalence independent of the diagram.

[I thought Jones used subfactors.]

I don't know anything about Von Neumann algebras. Certainly he didn't have a topological thing in mind.

People may have heard of Heegard Floer homology. This is another invariant of knots. This is geometric, in the spirit of gauge theory. The way you find this one, let me think for just one second about how to say this. Take a "perfect" Morse function on S^3. This means that the value of a critical point is equal to its index. The picture is that you take 0, 1, 2, and 3, and the inverse image of, say, 1.5, is a surface of high genus. In the simplest case this is an S^2, there are no 1 or 2 index critical points. If people know the picture of the Hopf fibration, you have a foliation of S^3 with circles with the one cross circle, and then a neighborhood of the circle is a torus. On the level set, there are the α-circles which come from the descending manifold and the β-circles coming from the ascending manifolds. This is the nicest possible way this can be. Now, if your knot is sufficiently complicated, you could have it be a flow line from 0 to 3 and back down from 3 to 0. You've traded the complication of the knot for the complication of the Morse function. Heegard Floer says, so, think of this as a twice-punctured surface of some genus, and take, call it Σ_g, take $Sym^g\Sigma_g$, what is this? This is $(\Sigma_g) \times \cdots \times (\Sigma_g)/S_g$. If you did this with something other than a Riemann surface, you get something badly singular, but if you take, say, $Sym_n(\mathbb{P}^1)$, you get $\mathbb{CP}^n$, and the same calculation says that this symmetric power is smooth, and in fact a symplectic manifold. The product of the α_i modulo the symmetric group is a Lagrangian submanifold, and similarly for the product of the β_i. Call them L_α and L_β. Let me then just say, Lagrangian Floer theory assigns a group to this, call it $HF(L_\alpha, L_\beta)$. This is Ozsvath-Szabo. I mention this one because it's almost the easiest one to define that is related to Khovanov homology.

I don't know who to attribute this to, but there exist spectral sequences from a variant of Khovanov homology to a variant of Heegard-Floer. It's shocking, usually, if you don't put in a lot of differentials, you don't get an invariant, here you are doing something geometric, you have more differentials than Khovanov. This turns out not to depend on the Morse function. There are many such Morse functions, but it's independent of which one I pick.

If you think about it, remember when I did Khovanov homology, we had $V = H^*(S^2)$ and then $(H^*(S^2))^n$. So L_α is $(S^1)^n$.

Seidel-Smith, and this is an interpretation, it's hard to know what they knew at the time, but I'll say due to Manolescu as well. To specify a Riemann surface, I can take $\mathbb{C}$, put some critical points, and look at the double branched cover with this branching locus. That's one way. The circles we'd be talxing about would be like taking the circles, it's a double branched cover, there are two points upstairs except at the critical points. So the inverse image of an interval between two branch points is a circle, called a matching cycle. So these are the α curves. Now it's not very hard to say how you'll change S^1 to S^2. Instead of talking about double branched covers, you want to talk about Lefschetz fibrations of a surface. How do I obtain the double branched cover? Fix p_τ a polynomial with roots $\tau_1, \ldots, \tau_n$. I can consider inside $\mathbb{C}^2$ the set $\{w^2 + p_\tau(z) = 0\}$. The projection to z makes this a double branched cover branched at τ_i, since this will have two roots in w unless $p_\tau(z) = 0$.

In the next dimension, take, in $\mathbb{C}^3$, the surface $\{u^2+v^2+p_\tau(z) = 0\}$. Now it's an affine algebraic surface and it projects down to $\mathbb{C}$ with no critical points except for the roots of τ. The smooth fiber is a cylinder, a conic. If the value is nonzero, that's a conic, topologically a cylinder. What happens if I draw the matching cycles? The fiber over the bad points is a collapsed cylinder, a double cone. This collapses a cycle, the vanishing cycle. So the vanishing cycle on both ends of the branching is the same, since there's only one circle on the cilinder. The result of gluing together is then a 2-sphere, which is again called a matching cycle. We've upgraded Heegard Floer homology to something using 2-spheres. The next step would be be to do the symmetric product. Call this fibration

[Owen: we put the punctures associated to the knot. How did that show up?] You compute it in the symmetric powers of the punctured thing. Call it A_n and consider the Hilbert scheme $Hilb^n(A_n)$, which is a resolution of the $Sym^n(A_n)$. Don't trust my numbering, there's standard numbering I could have messed up. The idea is to find an invariant of knots by studying Lagrangian Floer homology inside this Hilbert scheme. The truth of the matter is, even though the Hilbert scheme is smooth, computing Floer homology there is not known to be doable over, if you don't use a power series ring. In practice they consider modding out by divisors, but actually they use a Lie-theoretic interpretation.

I should stop this part.

[Is there a Morse thing here?]

We'll just work with the surface. There are other ways to understand the knot from the point of view of the double branched cover. The moral of the story is there's the part living above and below the S^3, and that will correspond to drawing part of the diagram above and below. Each piece will give a Lagrangian and the invariant will be Floer homology between these two.

[Where is the filtration?]

It's hard to see. There are small curves that live near the intersection. Those give rise to Khovanov homology. The big ones give you the higher differentials. It's a funny looking spectral sequence to make the Khovanov homology as E_1. You can't use this to show that Khovanov is invariant.

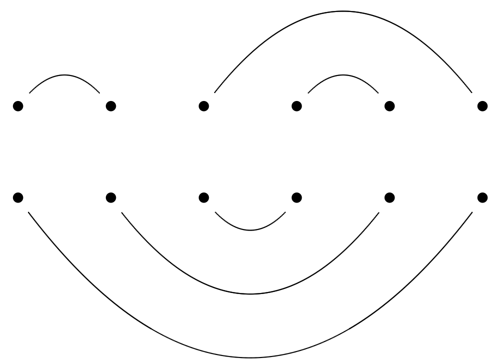
[Is there a field theoretic expectation for why there would be a connection?]

Writing down a field theory formula shouldn't make sense. Essentially the only one we know is Floer homology. If it has such an interpretation, it should be as a Floer homology.

I think, not for this sequence, one of the problems is, are the other pages invariants of knots? There is some work in this direction.

2. Seminar
Symplectic Khovanov homology and constructible sheaves

Thank you. I noticed that I only have one tiny little piece of chalk. I'll start at the beginning. I tried to put context in the pretalk. Let's start with an embedding $S^1 \hookrightarrow S^3$. In the usual way, you can project to $\mathbb{R}^2$ and get a knot diagram [scribble. Laughter.] It's pretty well-known that you can present any such knot diagram as a braid closure. For this talk we will always have an even number of points on the line, and instead of a hideous mess, you have strands, they cross one another, and so on [picture]. This is a braid. A braid closure, you can imagine something living entirely above this line and entirely below this line, and there will be no crossings. I may produce a knot, maybe a link, that's okay. So I could use different crossingless matchings. One way of thinking of Khovanov homology is to construct a diagrammatic algebra called the arc algebra. At this university, we would call this the arc category. There are other places where people prefer algebras. This will have as objects crossingless matchings. Then the next step will be to define morphisms in this category. Take a crossingless matching such as this one, and I want to take Hom of this with some crossingless matching. So call the first one M_1 and the second M_2, I'll draw that one below the line.



Now $Hom(M_1, M_2)$ will be $V^{\otimes \#\text{components}}$. Now I must tell you what V is, it's H^*S^2. There are compositions that are done diagramatically.

Khovanov constructs a functor on the category of complexes associated to each braid γ. So $\gamma \rightsquigarrow T_\gamma$. I will not define this now, I'll give you a way of thinking about it that is conjecturally the same, and it's an endofunctor of twisted complexes over the arc category. Think representable functors from Arc_n to Arc_n. The triangulated closure of this category. You could take all modules over this category, functors from Arc to $Vect$. There's a small part of this which are built from the images of the Yoneda modules. There is a functor from Arc to $Funct(Arc, Vect)$, and it's the image of this.

Now the Khovanov homology of a knot which is $M_1\gamma M_2$ is $Hom(\gamma M_1, M_2)$ which is a categorical representation of the braid group.

This does not depend on M_1, M_2, or anything else. The proof goes through, there's no clean proof.

If I give you two different braid presentations, there are a finite set of combinatorial moves that give you the same presentation, you can use Markov moves, I guess you could check invariance there.

I find this distasteful. Let's do some geometry. The piece of geometry I want to start with, I want to think of these points, given $2n$ distinct points τ_1 through τ_{2n}, I get a polynomial which has roots at these points, call it p_τ. The points are in $\mathbb{C}$. This polynomial, I can use it to do some geometry. Let me say what I want to do, which is to consider a surface X_τ and this will be the set of points $(u, v, z) \in \mathbb{C}^3$ so that $u^2 + v^2 + p_\tau(z) = 0$. The most interesting thing about this surface, people who know things should recognize this as an A_{2n-1} surface, the number might be wrong, in the ALE classification. So it has a hyperKähler structure and very nice geometry. I want to say a little about how to think about this. I can project this using the z coordinate to $\mathbb{C}$ and ask what is the fiber. The critical points are exactly $\tau_1, \ldots, \tau_{2n}$, because this squares to zero, those are the only critical points. The fiber is a conic, which I will draw like this cylinder, $u^2 + v^2 = \lambda$. I'll draw some points here, and the fiber over a random point is one of these cylinders, and the fiber over a point has a circle pinched. Then I can associate to a path between thees an S^2, which is actually a Lagrangian. Already we can do some geometry. Before a crossingless matching was a combinatorial thing; now this is n Lagrangian spheres in that ALE surface.

[Have you ordered your zeros?]

Yes, I did that to start. I introduced the τ to get the braid group action. Let me finish what I'm saying. You have n Lagrangian spheres. They are disjoint. The product is a Lagrangian $(S^2)^n$ inside $(X_\tau)^n$ which projects down to $Sym^n(X_\tau)$. These don't see the diagonal when you project down, and the symmetric product has a resolution, the Hilbert scheme, and your product embeds there in $Hilb^n(X_\tau)$. No one calls it this, but there is something called the symplectic "arc category" which is the full subcategory of the Fukaya category of $Hilb^n(X_\tau)$ with these Lagrangians as objects.

What is essentially known is that this category at the cohomological level is the same as Khovanov's algebra, the arc category. The question that I would like to answer that I would like to answer is, is this true as A_∞ categories? If it were true there, one would be able to conclude that all of Khovanov homology was detected by this symplectic construction.

Now let me note that as τ varies in the configuration space of $2n$ points in $\mathbb{C}$ we get a family of symplectic manifolds, all of which are symplectomorphic. Therefore we get a family of Fukaya categories. As you go around a loop in the configuration space, this is essentially the same thing, $\pi_1 Conf_{2n}(\mathbb{C}) \cong Br_{2n}$. Then you get from γ in the fundamental group the functor $T_\gamma : Fuk(X_\tau) \to Fuk(X_\tau)$. Now I can define symplectic Khovanov homology of a knot to be, take Floer homology of one of these Lagrangians, so that $L_1, \phi_\gamma L_2$ where L_i is the Lagragian conrresponding to the crossingless matching M_i. These are now described by a family of symplectic manifolds over configuration space.

So you can state this as being, is this symplectic arc category formal, since the Khovanov category has no differential.

There is a lie, because I can't quite use the Hilbert scheme, for people who know this, the ALE surface was an affine variety, so I have no closed holomorphic curves, the form is exact, there are such curves on the Hilbert scheme [missed a little] so insted there is the Lie theory approach, which is actually what Seidel and Smith did. How do I want to say this? Let's start with sl_{2n}, and I'll be very concrete, let me start with e which is

$$\begin{pmatrix} 0 & I & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

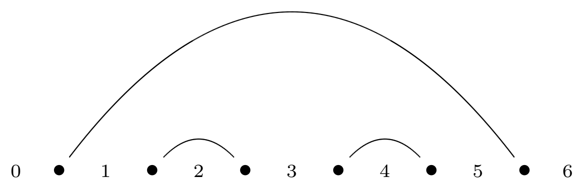
where everything is a 2×2 block.

Now S_e is the Springer fiber of e. It sits in flags in $\mathbb{C}^{2n}$, it's the flags so that $eF_i \subset F_i$.

If I pick coordinates $x_1, \cdots, x_n$ and $y_1, \ldots, y_n$, and e takes each one of these down by by one index.

Theorem 1. *(I don't know who this is due to)*
There exists a bijective correspondence between irreducible components of the Springer fiber and crossingless matchings.

I'm talking about this Springer fiber. So think that I have a crossingless matching



The arcs tell me that $e^{-1}(F_1) = F_3$, that $e^{-1}(F_3) = (F_5)$, and $e^{-3}(F_0) = F_6$.

Then the intersection between two components is controlled by S^2 raised to the number of circles in the glued diagram. If you look at the one example, choose F_1, then F_3 is determined. You have a $\mathbb{P}^1$ of choices for $\mathscr{F}_2$, and so this is a fibration of $\mathbb{P}^1$ but if you're like me and you're sloppy you can think of this as being a product of $\mathbb{P}^1$.

There is a space called the Slodowy slice, which is in fact a complex symplectic manifold which sits inside $Hilb^n(X_\tau)$ as an open subset, and includes the Springer fiber as a Lagrangian skeleton. Finally we're doing some symplectic topology. The intuition is that if you take any smooth manifold, I can look at its cotangent bundle, and somehow all the data about the symplectic manifold comes from the original manifold. There's a subtler sense, T^*X has a flow that contracts everything to the skeleton. The flow on the Slodowy slice contracts everything to the Springer fiber. So think of this as being like the cotangent bundle of this Springer fiber, which is not a manifold but isn't too far from being one.

Now we go a little bit faster. I said something about constructible sheaves in the title, now it's time for them to appear. I want to mention a fact due to Victor Ginzburg, who proved the following thing: consider the group, the Lie algebra first of all,

$$m = \begin{pmatrix} 0 & & & \\ & 0 & & \\ \vdots & \ddots & \ddots & \\ & \cdots & * & 0 \end{pmatrix}$$

where everything is 2×2 blocks. This isn't nilpotent because you have an extra row of zeros. It's in sl_{2n}. You can exponentiate to M the Lie group which acts on the flag variety $\mathcal{B}$ by diffeomorphisms hence by Hamiltonians on $T^*\mathcal{B}$. Whenever you have such an action, you have an induced map $T^*\mathcal{B} \xrightarrow{\pi} m^*$, so you wo have an induced map $\mathfrak{g}^* \to m^*$ which factors, you have a map $T^*\mathcal{B} \to \mathfrak{g}^*$. You want e to live in $\mathfrak{g}^*$, which I have to identify with $\mathfrak{g}$ using the trace. Call its image in m^* by the name χ. We should check that this is invariant under the coadjoint action. It is, so we can do Hamiltonian reduction $\pi^{-1}(\chi)/M$, and the result of Ginzburg is that this is actually the Slodowy slice. This is a specialization to $sl(2)$, something similar is true for the other lie algebras. In fact, you can do more, $\pi^{-1}(X) = Y_e \times M$. It's very late in the talk to be saying "now, here's my work." This is partially because this is work in progress (joint with I. Smith). What do you want to do? Naively, you'd like to observe that if you have such a situation, and $L \subset Y_e$ is a Lagrangian then $L \times M \subset T^*\mathcal{B}$ is also a Lagrangian. So this construction, remember, I had the symplectic arc category, which was a subcategory of the Fukaya category of Y_e. So we have something, a functor $Fuk(Y_e) \to Fuk(T^*\mathcal{B})$. Taking products in Floer theory is a subtle matter, but there is technology due to Mau-Wehrkeim (?)-Woodward which produces such a functor. Which Fukaya category of the cotangent bundle to the flag variety should we consider? This is not compact in the cotangent direction. When you have ends, you have lots of choices, will you do relative theory, or whatever? Here it's between something I call the Nadler-Zaslow category, and something I call the wrapped Fukaya category. In the latter, all objects vanish, so it should be the former.

There are some technical points which are why things are not proved, so ignoring those technical points, what you in fact get, noticing that M is contractible, crossing with the same contractible thing everywhere, this map of Fukaya categories is a fully faithful embedding. $HF^(L_1 \times N_1, L_2 \times N_2) \sim HF(L_1, L_2) \otimes HF(N_1, N_2)$, but this latter is just $\daleth$. Now we can appeal to Nadler-Zaslow which say there is a functor $Fuk(T^*\mathcal{B})$ to the derived category of constructible sheaves on $\mathcal{B}$ itself, which is an equivalence. So I've embedded this story in sheaves on $\mathcal{B}$.

There are lots of question I'd like answered. I can probably prove that these are what David tells me are called "tilting sheaves."

Questions.

(1) We started with an invariant of knots. Is there a direct proof that sheaves on $\mathcal{B}$ "contains" a [this] knot invariant.
(2) Can one prove formality of the symplectic arc category using the embedding into $Sh(\mathcal{B})$. There's so much more structure, these are sheaves. You could do things locally, there's a weight decomposition floating around, and usually when you have an extra grading you have a good shot at proving formality.

Thank you.

[Why is everything trivial mapping to the wrapped Fukaya category?] Let's look at $T^*\mathbb{R}$. [Picture] There's [unintelligible]that gives rise to this Lagrangian. Any object in the wrapped Fukaya category that can be perturbed to not intersect the zero section goes away. That's basicall what happens here.

[We had powers of S^2 floating around. Did the embedding depend at all on any matching having to do with the ordering you chose on τ? Would you get a different skeleton if you chose a different order?] The skeleton is independent of choices, but the presentation will depend on this.

[Missed a question or two]

[These happen to be constructible. What else do you know about the image of this embedding?]

There are things I can say but they are long, so let's talk later. It's not the Brouhat (?) stratification.

[You mentioned at the beginning the Hilbert scheme. [unintelligible]hyperK ahler? Could you use coherent sheaves?] [unintelligible]constructed Khovanov homology with coherent sheaves, or almost. [Missed some answer]. This has essentially been done, but proving mirror symmetry...