COLLOQUIUM
YURI MANIN
MORDELL-WEIL PROBLEM FOR CUBIC SURFACES: A COMBINATORIAL APPROACH

GABRIEL C. DRUMMOND-COLE

Thank you. I will divide this improvised talk into three parts:

(1) The usual Mordell-Weil for cubic curves
(2) What do we know about its 2-dimensional analog?
(3) A kind of new approach that goes via combinatorics but falls short of providing an answer we'd like to get

You have an elliptic curve or finitely generated extension of $\mathbb{Q}$, $[K:\mathbb{Q}]<\infty$, and the it's a torus and the resulting group $C(K)$ is an Abelian group. Mordell-Weil says that if this is nonempty, then it's finitely generated.

A geometric statement that allows generalization is that if it is nonempty, then $C\subset\mathbb{P}^2_K$, an embedding as a cubic curve which has a relation of colinearity $\mathcal{L}\subset C(K)\times C(K)\times C(K)$, that is $(x,y,z)\in\mathcal{L}$ if and only if $x+y+z$ is the intersection of C with a line. This is obviously S_3-symmetric so it defines a binary composition law: $x\circ y=z$ means that $(x,y,z)\in\mathcal{L}$. To get an Abelian group from it, you choose a $u\in C(K)$ and define $x+y=u\circ(x\circ y)$. Then you get an Abelian group with identity u.

How the usual proof of finite generation proceeds in this context: the basic statement, let's say in the case $K=\mathbb{Q}$, so $x=(x_0:x_1:x_2)$ where these are pairwise prime. I'll define $h(x)=\sum|x_i|$. The main step, or practically the end of the proof is, you prove that there is a constant so that whenever $h(x)$ is larger than this constant, then there are y and z so that $x=y\circ z$ and $h(y)$ and $h(z)$ are less than $h(x)$. Every point of sufficiently large height is decomposable. Then finite generation follows rather easily. Take all the points of lower height. There are finitely many of them.

In order to prove this, you do two different steps. You first prove the same story but in the case when x is the third intersection point of a tangent, $x=y\circ y$.

Then you prove "weak Mordell-Weil," that modulo such things everything is finite, $C(K)/2C(K)$ is finite. [I missed something.]

Two more remarks. It got recognized pretty late that, well, this is an unnormalized height. If you want something quantitative, a much nicer statement is true. You can define $\hat{h}(x)=exp(\lim_{n\to\infty}\frac{1}{n^2}h(nx))$. This will be the Euclidean length on $C(K)\otimes\mathbb{Q}$.

A corollaray is that the number of points in $C(K)$ with $\hat{h}(x)\leq H$ is something like a constant times $(\log H)^{rkC(K)/2}$.

The second remark is, if you allow singular curves, then Mordell-Weil stops being true. You don't get anything finitely generated. In the case of a [unintelligible]you get K^* and with a cusp you get K_+. Neither, of course is finitely generated in this case, so for them it's not true.

Now let's pass to cubic surfaces for simplicity, V, in $\mathbb{P}^3$. Let's follow step by step and see what we can transport to this situation. So $\mathcal{L} \subset V(K) \times V(K) \times V(K)$ is a triple relation unless there is a K-rational line in V. If K is sufficiently large there are 27 lines in V. If (x, y, z) lie on a line in V you could make a choice that they belong to $\mathcal{L}$, or don't. But if you're dealing with a minimal surface, then this is a perfectly nice definition.

Now if you try to use $\mathcal{L}$ to define the composition law, then the composition law stops to be well-defined when $x = y$ because a tangent line is not defined uniquely. You have a lot of possibilities. Take $T_x V$ and you can put your line in any direction. The set of points you get will be the two bad cases K^* and K_+.

Now next, what about the definition of addition. If there is no trouble with $x = y$, it is well-defined. You can define a partial symmetric composition law, or I could say let $x \circ x$ be any of the rational points. This leads immediately to two cases, both conjectural [I missed an aside.] There is a strong Mordell-Weil: there exist a finite number of points x_k so that for any $y \in V$ there exists a non-associative word so that whenever you specialize $X_a \to x_a$ and restrict to the case where there are never compositions $z \circ z$, then you will get y.

The weak version, everything is the same, but instead of restricting so you never get $z \circ z$, you allow $z \circ z$ to mean anything in $T_x V \cap V(K)$.

If such a finite generation holds we'll call it weak Mordell-Weil. Modulo tangent lines it's finitely generated.

So there are two versions. There are some paradoxical cases where we know we have weak Mordell-Weil. If K is algebraically closed, then we have weak but not strong.

There are conjectural asymptotics. Wherever we had a power of a logarithm we will have instead [unintelligible].

Now I pass to some numerical evidence. There are several numerical experiments related to this conjecture. Let me describe some of them. Here is selected data. Consider the simplest case, that I considered about 40 years ago with weak computers. Consider $[1, 2, 3, 4]$, the surface $x_0^3 + 2x_1^3 + 3x_3^3 + 4x_4^3 = 0$. For weak Mordell-Weil, it looks like mybe it's generated by $p^0 = (1 : -1 : -1 : 1)$. The first 15222 points are generated by this one (weakly). The first point that was not shown to be generated, the first bad point has height $23, 243$. You are trying to organize a computer experiment to get results like that, you have to ask yourself, how do you do this?

There exists a small but stable percentage of points that do not want to be decomposable in this particular way. This seems to be linear. How can weak Mordell-Weil hold? you consider words of longer length. The maximal length of non-associative word you need to check the first $15, 222$ is thirteen. So it looks like you should use words of larger and larger length. Then this means that our classical arguments won't work. They first go far away, have very large height, then they come back. That's very funny and makes it unclear why finite generation should hold at all. But it looks like it holds. He did this for several diffferent cubics. [Missed a little.]

There seems to be no theoretical connection between the numerical evidence and [unintelligible].

Now I will briefly pass to the third part, combinatorics. This struck me sometime in the 90s. There is a classical result from which finite generation follows. The context is pretty silly. Replace V with the projective plane $\mathbb{P}^2$. The generation of rational points, you know them anyway, but still, what you do, start with a finite number of them and draw all lines through pairs. Then take intersection points. They will be rational. Start drawing lines, and again add the intersection points.

Question: for K finite over $\mathbb{Q}$, is it is sufficient to start with a finite amount? Yes, but you prove it in a weird way. You consider $\mathbb{P}^2(K)$ as a combinatorial object, a set and the set of subsets that are called lines. This $P, \{\ell\}$ has well-known properties. When is such a thing come from a K. The answer is given by the famous theorem that will now become and axiom, due to Pappus.

Consider six points. Choose any cyclic order. It might happen that a diagonal and the corresponding two sides intersect in a point. Then it might be true for another diagonal. Then it is true for the third one.

It was found later that if something satisfies the Pappus condition then it comes frem a field. It was a start of the modern model theory. This tells you that two things thought of as structures are equivalent in a nice and nontrivial way. How from this does it follow that you have finite generation. The argument runs as follows. [An argument, I couldn't follow.]

So then the question is, can we do this for cubic surfaces. For some time I couldn't see how to start. Then I devised some way of doing this, in my arXiv paper of January 1, 2010. I would like, first of all, to take something that looks like a combinatorial cubic surface. It should be a set of points W, along with a colinearity relation $\mathcal{L} \subset W \times W \times W$, and then I want it to be symmetric, that's easy. I'll also need $\{P\}$, a coplanarity condition. What I want is that if we add a list of combinatorial properties that would allow me to show that there exists a field K and a cubic surface V over K and an embedding $W \subset V(K)$ under which colinearity and coplanarity hold. I could not pass back. How could you prove that $W = V(K_0)$ or something like that.

Very briefly, how do I do it? For a long time I didn't know how to start, but then I realized that from the singular curves I could try to put them togother to construct the old C. The axioms should include tangent combinatorial planes. Again everything is combinatorial. One group is conjecturally C_* and the other C_+. In the first approximation, everything is easy to do. I can get a birational correspondence by considering two points if the respective tangent lines intersect at one point here [picture and description]

The trouble is that the line itself is not in W. In the geometric case that's okay but I want to avoid this. To do so I use the coplanarity condition. Instead of imagining i know all points and the line, I imagine I know the intersection of V and W with the plane containing these lines

So the list of conditions become axioms that hold in the geomtric case, and we get this result. But I challenge how to go there. Thank you.