GEOMETRY/PHYSICS SEMINAR

GABRIEL C. DRUMMOND-COLE

1. Si Li
Feynman diagrams, perturbative field theory, and Calabi-Yau

I want to start with something very elementary, so, calculus. I'll start with Feynman diagrams. I want to consider $Z_\lambda = \int_{\mathbb{R}^n} d^n \times e^{-\frac{1}{2}\sum x_i A ij x_j} e^{\frac{\lambda}{3!}\sum V_{ijk}x_ix_jx_k}$, where A_{ij} is symmetric positive definite matrix. This definiteely does not converge so we'll consider it as a formal power series:

$$\int_{\mathbb{R}^n} d^n \times e^{-\frac{1}{2}\sum x_i A ij x_j} \sum \frac{1}{m!}(\frac{\lambda}{3!}\sum V_{ijk}x_ix_jx_k)^m$$

We'll intoduce

$$Z[J] = \int_{\mathbb{R}^n} d^n \times e^{-\frac{1}{2}\sum x_i A ij x_j + \sum J_k x_k}.$$

Then

$$Z_\lambda = e^{\frac{\lambda}{3!}\sum v_iv_jv_k\partial_i\partial_j\partial_k Z[J]}|_{J=0}$$

but we can also write

$$Z[J] = \int_{\mathbb{R}^n} d^n \times e_i^{-\frac{1}{2}\sum (x-A^{-1}J} A_{ij}(X - A^{-1}J)_j e^{\frac{1}{2}\sum A_{ij}^{-1}J_iJ_j} = Z_0 e^{\frac{1}{2}\sum A_{ij}^{-1}J_iJ_j}$$

So we get

$$\begin{aligned} Z_\lambda &= Z_0 e^{\frac{\lambda}{3!}\sum v_iv_jv_k\partial_i\partial_j\partial_k} e^{\frac{1}{2}\sum A_{ij}^{-1}J_iJ_j}|_{J=0} \\ &= Z_0 e^{\frac{1}{2}\sum A_{ij}^{-1}\partial_i\partial_j} e^{\frac{\lambda}{3!}\sum v_{ijk}J_iJ_jJ_k}|_{J=0} \end{aligned}$$

So we look at the λ^2 term $(\frac{1}{2}\sum A_{ij}^{-1}\partial_i\partial_j)^3(\frac{\lambda}{3!}\sum v_{ijk}J_iJ_jJ_k)^2$.

We look at propagation. We have A_{ij}^{-1} which we picture as $i \overset{A_{ij}}{\text{———}} J$ and the trivalent vertex λv_{ijk}. So we get for the λ^2 term two options, the dumbell and the theta. So we get for the dumbell $\lambda^2 v_{ijk}v_{i'j'k'}A_{jk}^{-1}A_{ii'}^{-1}A_{j'k'}^{-1}$, and we have to divide by 8. For the theta graph, we get $\lambda^2 v_{ijk}v_{i'j'k'}A_{ii'}^{-1}A_{jj'}^{-1}A_{kk'}^{-1}$, and we divide by 12. So if I call the graphs Γ_1 and Γ_2, this is $\frac{W(\Gamma_1)}{|Aut\Gamma_1|}$, which is where 8 and 12 comes from.

Any questions? You should convince yourself after some calculus, that Z_λ can be written as

$$Z_0 \sum_{\text{trivalent diagrams }\Gamma} \frac{W(\Gamma)}{|Aut\ \Gamma|}$$

You have to do some calculation to see this. This includes disconnected graphs as well, all possible graphs. Physicists want only connected graphs; then this is $Z_0 exp(\sum \frac{W(\Gamma)}{|Aut\ \Gamma|})$ where this time the sum is over disconnected graphs. Call this $Z_0 exp(F_\lambda)$ and F_λ is the free energy.

Now I want to use Feynman diagrams to constuct some invariants. Any quesitons before I proceed? Everybody is happy because of this formula?

Let's consider and infinite dimensional example, this is Chern-Simons theory. The geometric data is the following. Suppose M is a compact three dimensional manifold. Let g be the Lie algebra of a Lie group G and we'll consider only trivial G bundles. Connections will be one forms valued in g, so $\Omega^1(M, g)$. We have the interesting action $CS[A] = \int \frac{1}{2} Tr\ A \wedge dA + \frac{1}{3} Tr\ A$. Suppose a basis $\{g_a\}$ of the Lie algebra, orthonormal with respect to the Killing form, and $Tr\ g_a g_b = \delta_{ab}$. Then I can write $A = A^a g_a$, and then I can write this as

$$\int \frac{1}{2} A^a \wedge dA^a + \frac{1}{3} \sum f_{abc} A^a \wedge A^b \wedge A^c.$$

Physicists would tell you to look at $\int DAe^{CS[A]}$, they want to integrate that along the space of connections $\Omega^1(M, g)$, but this is ill-defined. Okay, so problem one is that this is an infinite dimensional space. But for physicists, they can play with this guy.

Problem two, there is a gauge symmetry $A \to uAu^1 + udu^{-1}$ for $u : M \to G$.

What are we going to do with this one? We want to actually, completely mimic the finite dimensional case. We have two terms, why not do the same thing for the Chern Simons action as we did before. Let's use Feynman diagrams. First we need to know the propagator. The propagator, there is a d here, so what about d^{-1}? Definitely not a well-defined operator. We'll have to introduce a gauge-fixing condition. We'll replace d^{-1} with $\frac{d^*}{\Delta}$, pick a metric and this is well-defined. Then $dd^* + d^*d = \Delta$. When the Laplacian is well defined this is called the Green kernel, and this is in $\Omega^2(M \times M \backslash \Delta, g \otimes g)$. This is singular on the diagonal. This can be used for the propagator. What about the vertex? The vertex with A_1, A_2, and A_3 is $TrA_1A_2A_3$.

Okay, now I have a propagator. So, let's consider the following: consider two-loop invariants, the invariants we just discussed. We'll write down what the diagrams mean. The barbell is

$$\frac{1}{8} \int_{M \times M} Tr\ L(x_1, x_1) \wedge L(x_1, x_2) \wedge L(x_2, x_2)$$

and the theta is

$$\frac{1}{12} \int_{M \times M} Tr(L(x_1, x_2)^3).$$

We want to define something like this, but we have problems when x_1 goes to x_2 along the diagonal. For the Chern-Simons case, this is finite. In general this is divergent.

The second one, problem 4, we want to define invariants out of this integration, and

$$CS[A] = \int Tr(\frac{1}{2} A \wedge dA + \frac{1}{3} A^3)$$

and this doesn't involve a metric. We used the metric to define d^* and the kernel L. So this seems like a metric construction. So the next problem is the (in)dependence o metric (also called gauge fixing). There are a bunch of problems here. To solve the divergence problem, the physicists say that we can add additional terms to cancel the singularity. This procedure is called renormalization.

So problem one will be solved with Feynman diagrams, problem two with the BV-formalism (gauge fixing), problem three with regularization (renermalization), and problem four with the quantum master equation. Any questions? So far, so good?

I will discuss another example less known to mathematicians.

[unintelligible]Calabi-Yau, there's a gauge theory, called Kodaira-Spencer. Suppose X is a Calabi-Yau 3-fold, a 3-dimensional complex manifold with a nowwhere vanishing holomorphic 3-form Ω. We want to consider the following space:

$$\Omega^{0,1}T_X^{1,0}$$

This is going to be the space where our fields will live. There is a natural identification of this space with differential forms. We can take the contraction of this guy: $\Omega^{0,1}T_X^{1,0} \overset{\vee\Omega}{\to} \mathscr{A}_X^{2,1}$.

So now consider $H^{0,1}_{\bar\partial}(T_X^{1,0})$, sometimes called $H^1(T_X)$. This is the tangent space of the deformations of complex structures of X. Let's consider, we need to consider a smaller subspace of this guy. Let's call this $\mu \in \Omega^{0,1}(T_X^{1,0})$ so that $\mu \vee \Omega$ is in the image of ∂. The set of such μ sits inside $\Omega^{0,1}(T_X^{1,0})$. We want to consider the following action, called the Kodaira-Spencer action:

$$KS_*[\mu] = \frac{1}{2}\int (\mu \vee \Omega) \wedge \frac{1}{\partial}\bar\partial(\mu \vee \Omega) - \frac{1}{6}\int \Omega \wedge ((x+\mu)^3 \vee \Omega).$$

This won't depend on the choice of preimage under ∂. This is the quadratic part, very strange, right? My μ lies in the space I just described.

Physically we want to look at [unintelligible]. If we look at the variation $\frac{\partial KS[\mu]}{\partial \mu}$ well discover that $\bar\partial(x+\mu) + \frac{1}{2}[x+\mu, x+\mu] = 0$ so $x+\mu$ defines a new complex structure on X! This is interesting right? The critical locus gives us the moduli space of complex structures.

What do we expect from this structure? Later in my talk I'll replace the condition with some formal variables. So, first, suppose we have the moduli space M of complex structures on X. At every point in the moduli space we can define an action. It seems like we can find some invariants F_g living on the moduli space. [unintelligible]mirror symmetry. It's a very strange object. From a mathematical point of view it's not known how to do this in general.

[Ezra: what's "this"?]

How to construct F_g rigorously in the B-model.

[Ezra: I thought they have a perturbative expansion.]

[Some discussion. The individual Feynman diagrams are divergent here for higher genus.]

Some perspective for the propagators. The propagators, we want to invert our operator, so it will be $\partial\frac{1}{\bar\partial}$. How do I do that? Pick a metric, and replace this with the kernel

$$\frac{\partial\bar\partial^*}{\Delta} = \int_0^\infty \partial\bar\partial^* e^{-t\Delta}dt.$$

[Ezra: Do the diagrams have different properties if it is Calabi-Yau?]

I hope that a lot of cancellation will happen. It's a complicated calculation.

[Ezra: Does this give insight into the existence or construction of Calabi-Yau metrics?]

The answer is, I don't know.

2. BCOV theory on the elliptic curve and higher genus mirror symmetry

So I'm going to discuss this topic, joint work with Kevin Costello. I'll discuss background, BV quantization of gauge theory, BCOV theory on elliptic curves, and then mirror symmetry.

Let me talk about some background. Consider a pair of Calabi-Yau three-folds, a so-called mirror pair X and $\tilde{X}$, and physicists predict by string theory some invarinats. Call these F_g^A and F_c^B. So these two data can be identified, so what do we know about these invariants? These denoe the A-model and B-model in physics. The A-model is mathematically constructed in all genera via the Gromov-Witten invariants, counting maps from a genus g curve to your Calabi-Yau.

What about the B-side? We know the genus zero case. This is known as the string potential or the [unintelligible]coupling. People can check mirror symmetry for genus zero. Essentially thy proved that $F_0^A = F_0^B$ for a large class of Calabi-Yaus by Givental and also Lian-Liu-Yau.

What about higher genus? What about F_g^B? Actually, the geometric meaning is not very clear. We don't have a very good mathematical construction. It would help to establish mirror symmetry for higher genus to know what this is.

In the early nineties there was a breakthrough paper by Berghesky-Cecotti-Ooguri-Vafa, a physics paper, they proposed a gauge theory on a Calabi-Yau three-fold, and what they proposed was the following. It's a sort of Kodaira-Spencer gauge theory. Ther proposed a quantization of the Kodaira Spencer should give F_g^B for all g. This is now called BCOV theory.

There were a lot of physics breakthroughs after this paper. The holonorphic anomaly equation was used to [unintelligible]. Then they predicted that F_1^B is given by analytic torsion. There is a proof a couple of years ago. Yamaguchi-Yau showed that there are polynomial relations, recursive relations for F_g^B that they used to compute these for $g = 2, 3, 4$, something like that. Then Huang-[unintelligible]-Quackenbush computed F_g^B for $g \leq 51$. These last two were done for the mirror quintic.

This is a brief history.

The main result is, we quantize the BCOV gauge theory on the elliptic curve E and from this one we can define $F_g^B(E)$ for all g and can check that $F_g^A(\tilde{E}) = F_g^B(E)$. This works in the sense of mirror symmetry.

[Can you do this for any dimension?]

We can do it in 3 and 1. There will be problems of divergence.

Let me discuss a little bit about BV quantization of gauge theory. This has some meaning to physicists, but there's a geometric realization in terms of very geometric data. Let me specify some gauge data. We need $\mathcal{E}, Q, Q^{GF}, K_L, S$. The first thing $\mathcal{E}$ is a space of fields, that we should think of as maybe the sections of a vector bundle. It's a graded complex. Then Q is a differential of degree 1 on $\mathcal{E}$ and Q^{GF} is degree -1 and their commutator is a generalized Laplacian Δ. You need to use the fact that the heat kernel is well defined for this operator. Then K_L is the regularized BV kernel that I will explain later, and S is the classical gauge action.

Example 1. *[Chern-Simons]*
Here X is a compact 3-dimensional manifold and g is a Lie algebra. We'll consider for simplicity

$\Omega^*(X,g)[1]$ *(shifted) as my space of fields* $\mathcal{E}$. *My differential is* d. *To define* Q^{GF} *will be* d^*, *picking a metric. Then* Δ *is the ordinary Laplacian. Then* K_L *is the heat kernel* e^{-L_Δ}. *The action is the classical Chern-Simons action* $S[A] = \int \frac{1}{2} TrA \wedge dA + \frac{1}{3} Tr\ A^3$.

From this theory we will define some invariants usiong Feynman Diagrams. For example, we'll need propagators and a vertex. The propagators are given by inverse of the quadratic part of the action, so we replace d^{-1} with the propagator $P(x_1, x_2)$ the kernell representing the operator $\frac{d^*}{\Delta} = \int_0^\infty d^* e^{-t\Delta} dt$

The vertex is given by $A_1 \otimes A_2 \otimes A_3 \mapsto \int Tr\ A_1 \wedge A_2 \wedge A_3$. Then on the theta graph we get $\int_{X \times X} Tr(P(x_1,x_2)^3)$. But as a problem, $P(x_1,x_2)$ is singular as $x_1 \to x_2$. We need to talk about effective field actions. So we consider the truncated propagator

$$P_\epsilon^L = \int_\epsilon^L Q^{GF} K_u du$$

which in this case is $\int d^* e^{-t\Delta} dt$.

Let me call $\mathscr{O}(\mathcal{E})$ the space of functionals and $\mathscr{O}_{loc}(\mathcal{E})$ the space of local functionals, which means that this is integration of some Lagrangian.

Definition 1. *A family of functions* $I[L]$ *for* $L \geq 0$ *so that* $I[L] \in \mathscr{O}(\epsilon)[[\hbar]]$ *is said to satisfy* renormalization group flow *if the following is true:* $I[L] = W(P_\epsilon^L, I[\epsilon]) = \sum_{\Gamma\ connected} \frac{W_\Gamma(P_\epsilon^L, I[\epsilon])}{|Aut\ \Gamma|}$ *for all* L *and* ϵ. $I[L]$ *defines a family of effective field action.*

For the Chern Simons theory, I would like to define

$$\int [DA] e^{CS[A]}$$

But actually, if we use Feynman diagrams, you can write this as

$$e^{W(P_0^\infty),CS)}$$

but this is singular in general.

The theorem is the following. This is known to physicists for a long time and proved for us by Kevin Costello.

Theorem 1. *There exist perturbations* $\sum \hbar^n S_n^{CT}$ *so that*

$$\lim_{\epsilon \to 0} W(P_\epsilon^L, S + S^{CT}(\epsilon)) = I[L]$$

exists.

These $I[L]$ *satisfy the renormalization group flow automatically.*

We start from a classical action S and do a regularization to define a family of effective actions $\{I[L]\}$. At the beginning we had to choose a metric. To define our integral and the propagators we needed a metric. What about the dependence of $I[L]$ on the metric? A priori Chern Simons doesn't depend on the metric. This is solved by the quantum master equation. Let me say a little bit about it.

Given $\alpha \in \mathcal{E}$, we can define a derivation $\frac{\partial}{\partial \alpha}$ on $\mathscr{O}(\mathcal{E})$ which takes I to $(\frac{partial}{\partial \alpha} I)[\beta_1, \ldots, \beta_n] = I[\alpha, \beta_1, \ldots, \beta_n]$. Similarly, $K_L \in \mathcal{E} \otimes \mathcal{E}$ leads to $\frac{\partial}{\partial K_L} : \mathscr{O}(\mathcal{E}) \to \mathscr{O}(\mathcal{E})$.

Definition 2. *The regularized BV bracket is defined to be* $\{S_1, S_2\}_L = \frac{\partial}{\partial K_L}(S_1, S_2) - (\frac{\partial}{\partial K_L} S_1) S_2 - S_1(\frac{\partial}{\partial K_L} S_2)$.

Remark 1. *If* $S_!$ *and* S_2 *are local functionals the* $\lim_{L\to 0}\{S_1, S_2\}_L$ *exists as a local functional* $\{S_1, S_2\}_0$.

There is a requirement I omitted. I require $S[A] = \langle A, \theta A\rangle + \underbrace{I[A]}_{\text{at least cubic}}$ satisfying the property, called the classical master equation, that

$$QI + \frac{1}{2}\{I, I\}_0 = 0$$

The physical meaning is very clear.

Remark 2. *The classical master equation is the same thing as gauge symmetry in physics.*

Because I want to do quantization, I want to explain the quantum master equation. A family of effective actions $I[L]_{L>0}$ is said to satisfy the quantum master equation if

$$QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar\frac{\partial}{\partial K_L} I[L] = 0$$

[Picture]

The quantum master equation is some kind of consistency condition of the gauge symmetry, which has to do with the independence of the gauge fixing. If you satisfy the quantum master equation, when when you vary your gauge fixing, [unintelligible]will vary in a very nice way.

I just want to say, we hope to quantize the theory in the following sense. We start with a classical action and want to find actions at varying scales. But in the previous case, we could always find local counter terms. Here there is an obstruction. Here, the classical action S leads via regularization to the family $\{I[L]\}_{L>0}$ and then there are obstructions to finding solutions of the quantum master equation.

Definition 3. *A quantization of the gauge theory* S *is given by a family of effective actions* $\{I[L]\}_{L>0}$, $I[L] \in \mathscr{O}(\mathcal{E})[[\hbar]]$ *so that they satisfy the renormalization group flow, the quantum master equation, a locality condition, and so that when* $L \to 0$ *and* $\hbar \to 0$, *this is the classical action* I

Okay. I should say actually, there are usually some obstructions. The space can be described as a cohomology:

Theorem 2. *(Kevin Costello)*
The obstrcution space of the quantum master equation is given by $H^1(\mathscr{O}_{\ell a}(\mathcal{E}), Q + \{I, \quad\})$. *The master equation ensures this squares to zero.*

Generally speaking, suppose we want to do a computation. We can find some quantization of this theory. This is what I mean by BV quantization of gauge theory.

Now let me describe BCOV theory on elliptic curves. I have to specify my gauge data. So E_τ will be the quotient of $\mathscr{C}$ by the lattice $\{1, \tau\}$, specifying the complex structure. We want to consider the space of all polyvector fields $\Omega^{0,*}(\wedge^* T^{1,0}_{E_\tau})[[t]]$ as our space of fields.

The differential will be, well, to define Q and Q^{GF}, we can use contraction for holomorphic one-forms, so $\vee dz$ is a map $\Omega^{0,*}(\wedge^* T_E^{1,0}) \to \mathscr{A}_{E_\tau}^{*,*}$. We'll use this to define ∂ and $\bar{\partial}$. Then $Q = \bar{\partial} + t\partial$, and $Q^{GF} = \bar{\partial}^*$. The kernel is given by $\partial e^{-L\Delta}$. The last thing is the gauge action, which is defined in the following way: $I = \bigoplus_{n \geq 3} I_n$ which is a map $I_n : Sym^n(\mathcal{E}) \mapsto \mathbb{C}$ The value of I_n on $\prod t^{k_i} \mu_i$ is $\binom{n-3}{\prod k_i} Tr(\prod \mu_i)$, where $Tr : \Omega^{0,1}(T_{E_\tau}) \mapsto \mathscr{C}$ by $\alpha \mapsto \int_{E_\tau} (\alpha \vee \omega) \wedge \omega$.

Theorem 3. *I satisfies the classical master equation $QI + \frac{1}{2}\{I, I\}_0 = 0$. This tells us we have a well-defined classical gauge theory.*

Remark 3.
$$\binom{n-3}{\prod k_i} = \int_{\overline{\mathscr{M}}_{0,n}} \prod \psi_i^{k_i}$$
which shows a connection between the classical master equation and the topological recursive relations.