TOPOLOGY SEMINAR
MEHDI KHORAMI
A UNIVERSAL COEFFICIENT THEOREM FOR TWISTED K-THEORY

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Thank you for coming. Sorry for not having a title on the web. I want to quickly define twisted K-theory without going into to much detail. If you look at the Picard group $Pic(X)$ of a compact space X, this is the multiplicative subgroup of $K(X)$ consisting of classes of line bundles. $Pic(X)$ acts on $K(X)$ via the tensor product: $L, V \mapsto L \otimes V$. This action is functorial, if you have a map $X \to Y$ then $f^*(L \otimes V) = f^*L \otimes f^*V$. This action is functorial, so I can talk about the classifying space for this group. The classifying space for line bundles is $BU(1)$, and then I'll say $BU(1) \times BU \to BU$, I will not say $BU \times \mathbb{Z}$. So this is a map $BU(1) \times BU \to BU$. So we want to capitalize on this action here. You can actually sort of realize this action at the point set level. You can come up with models for the zero space of K-theory. So Atiyah and Segal showed you can take that space to be the Fredholm operators on a complex Hilbert space.

Let F be the Fredholm operators on an infinite dimensional complex Hilbert space H then $PU(H) \times F \to F$, $PU(H)$ acts on F by conjugation. And $PU(H)$ has the homotopy type of $BU(1)$, that is, $\mathbb{CP}^\infty$.

So $K(X)$ is the homotopy class of maps $[X, F]$. You can write it as homotopy classes of sections of $X \times F \to X$. So the idea is to replace this trivial bundle with more interesting bundles. We have the action on BU, and so we need to come up with $PU(H)$ bundles and then come up with the orbit bundle. Take $P \xrightarrow{\tau} X$ is a $PU(H)$ bundle, then I can come up with $P \times_{PU(H)} F$ with fiber F over X. Then $K^\tau(X)$ will be sections of this bundle with fiber F over X.

What you are doing is twisting by these bundles, which are $PU(H)$ bundles. Then I am twisting basically by principal $\mathbb{CP}^\infty$ bundles. I don't want to define it that way. We can lift everything in the language of spectra. What I'm really doing, there is a map $\mathbb{CP}^\infty = BU(1) \hookrightarrow BU$, and what I'm doing is

$$BU(1) \times BU \longrightarrow BU \times BU \xrightarrow{\otimes} BU$$

At the level of spectra, I have $\Sigma^\infty_+ \mathbb{CP}^\infty \to K$, and then smashing with K I get $\Sigma^\infty_+ \mathbb{CP}^\infty \wedge K \to K \wedge K \to K$.

How do I classify $\mathbb{CP}^\infty$ bundles? They are maps $X \to K(\mathbb{Z}, 3)$. I can look at $\Sigma^\infty_+ P_\tau \wedge_{\Sigma^\infty_+ \mathbb{CP}^\infty} K$. The homotopy groups of this are the twisted K homology. These maps into $K(\mathbb{Z}, 3)$ are in bijection with elements of $H^3(X, \mathbb{Z})$.

The first spectral sequence that is going to converge to this is similar to the universal coefficients spectral sequence. So if I look at $\Sigma^\infty_+ P \wedge_{\Sigma^\infty \mathbb{CP}^\infty} K$, I can write it as $K \wedge \Sigma^\infty_+ P) \wedge_{K \wedge \Sigma^\infty_+ \mathbb{CP}^\infty} K$.

There are things to worry about here. Take any category of spectra and these things work. These are the best spectra that ever existed. This tells me that there's a spectral sequence, $Tor^{s,t}_{K_*\mathbb{CP}^\infty}(K_*P_\tau, K_*)$ which goes to $K^\tau_*(X)$.

[How do you define Tor?] Whatever way you want to, these are graded modules. [Some confusion.] The K_* is the coefficient group of the coefficient ring of K-theory.

What is the module structure? So if I look at this earlier map at the homotopy theory level I get a map $K_*\mathbb{CP}^\infty \to K_*K \to K_*$, which gives K_* a modular structure over $K_*\mathbb{CP}^\infty$.

You can actually compute this map. Well, $K_*\mathbb{CP}^\infty$ is spanned by β_i, which are dual to the cohomology classes. I'll put them all in degree 0. So what is K_*K? Well, K_*K injects in $K_*K \otimes \mathbb{Q}$ which is $\mathbb{Q}[u, u^{-1}, v, v^{-1}]$, and so the image is Laurent polynomials $f(u, v)$ so that $f(t, rt) \in \mathbb{Z}[t, t^{-1}, \frac{1}{r}]$ For example, in the image, $p_i = \frac{1}{i!}v(v-u)(v-2u)\cdots(v-(i-1)u)$, these generate almost everything. So $t^i\beta_i \to p_i$.

But of course, under the map $K_*K \to K$, you take u and v to the same place. If you have $(v-u)$ you will take it to 0. So β_1 goes to v. So for $i \geq 2$, $t^i\beta_i \to 0$. β_1 takes care of the line bundle action, which makes sense.

But, what, as I said, what you are actually doing is you're twisting K-theory by a $\mathbb{CP}^\infty$ bundle.

Okay, I have a $\mathbb{CP}^\infty$ bundle with space $Bspin^c$ over BSO. If I Thomify this bundle, I get a map of spectra $\Sigma^\infty_+\mathbb{CP}^\infty \to Mspin^0$. I can twist $Mspin^0$ in a similar way.

This means I can define twisted $spin^c$, well, $Mspin^{c,\tau}(X)$.

So this is a theorem of Hokins and Hovey. A map $Mspin^c \to K$ induces a map

$$MSpin^c_*(K) \otimes_{Mspin^c_*} K_* \to K_*(X)$$

What I wanted to do was put twists here and see if this is still an isomorphism. If the twist happens to be zero, then you recover the K-theory space itself. If you prove this for every τ (on both sides this is in $H^3(X, \mathbb{Z})$) you recover Hopkins and Hovey.

You expect that the problem would be easier at odd primes. There is a map $Bspin^c \to Bspin \times BU(1)$, and this map is a homotopy equivalence. But this couldn't be a map of H-spaces, this is not an H-space splitting, which would give a trivial $BU(1)$ bundle over $Bspin$. So this is not an H-space splitting. But at odd primes BSO and $Bspin$ are the same. For $p > 2$, $Mspin^c = MSO \wedge \Sigma^\infty_+\mathbb{CP}^\infty$. At odd primes the action of $\mathbb{CP}^\infty$ falls apart. Then $Mspin^{c,\tau}(X)$ is $MSO(P_\tau)$ and $Mspin^c_* = MSO_*(\mathbb{CP}^\infty)$. So if I start with $Mspin^{c,\tau}(X) \otimes_{MSpin^c_*} K_*$, this is $MSO(P_\tau) \otimes_{MSO_*\mathbb{CP}^\infty} K_*$, and you can see that this is $K_*(P_\tau) \otimes_{K_*\mathbb{CP}^\infty} \hat{K}_*$.

At odd primes I should not have any higher terms, or the differentials should kill everything. But the branching of Tor is exactly what happens. Everything in the E_2 page of the spectral sequence just vanishes. The theorem is for $s > 0$, $Tor^{s,t}_{K_*\mathbb{CP}^\infty}(K_*P_\tau, \hat{K}_*) = 0$. This happens to be true not just at odd primes but at all p. So twisted K-theory of X is just $K_*(P_\tau) \otimes_{K_*(\mathbb{CP}^\infty)} \hat{K}_*$.

It's disappointing, everything gets killed off, there is nothing to think about.

For general twisting it's more complicated.

So how do you go about proving such a thing, that the Tor groups vanish? If you look at $\hat{K}_*$, it's definitely not flat. All the β_i map to 0, but if you multiply by β_i, you get an injective map. It couldn't be flat.

If you look at the map $K_*\mathbb{CP}^\infty \to K_*\mathbb{CP}^\infty$ by multiplication by β_i, this is an injective map, so I should get an injective map tensoring with this but I get a 0 map. Also for some P_τ, the group K_*P_τ is not flat. For example, if $X = K(\pi, 3)$ then it's not flat.

So what is going on is, they still have no Tor, and the idea is the exact functor theorem idea. What we actually have here, these guys are very special modules, K-theory of something. They are comodules over K_*K. So let me write that down.

To use this comodular structure, that's what you do. The things that go on in there, it's different. Let me tell you what these mean. So K_*P_τ is K_*-mod which is a comodule over K_*K, so there is a map $K_*P_\tau \to K_*K \otimes_{K*} K_*P_\tau$. The thing I'm going to use, the pair (K_*, K_*K) form a Hopf algebroid and these guys are comodules over it. Let's say we have a Hopf algebroid L, W, then we have a coproduct $W \to W \otimes_L W$. You have two unit maps 1_R and 1_L from $L \to W$. You can write $MU = MU \wedge S \to MU \wedge MU$ or $S \wedge MU \to MU \wedge MU$, for instance. Then you have an augmentation $W \to L$.

Let $f : L \to R$ be a map of rings. We have a notion of exactness. f is called Landweber exact if $F_R : M \to R \otimes_L M$ is exact. Here I go from comodules over (L, W) to R-modules.

A map is called exact, you want to take a resolution and tensor with R. So there is a lemma that if you have a Hopf algebroid (L, W) and W is flat over L, then the map $f : L \to R$ is Landweber exact if and only if $f \otimes \eta_R : L \to R \otimes_L W$ is flat. So $L \xrightarrow{\eta_R} W \cong L \otimes_L W \to R \otimes_L W$.

This is a very nice observation and actually does the job in my case. Now, I said K_*P_τ is a K_*-module. So I am working with (K_*, K_*K), and I can actually make this stronger, into $(K_*\mathbb{CP}^\infty, K_*\mathbb{CP}^\infty \otimes_{K_*} K_*K)$. This is the Hopf algebroid I want to consider. I want to prove the map $K_*\mathbb{CP}^\infty \to \hat{K}_*$ is Landweber exact.

In the view of this lemma, I need to prove that the map $K_*\mathbb{CP}^\infty \to \hat{K}_* \otimes_{K_*\mathbb{CP}^\infty} (K_*\mathbb{CP}^\infty \otimes_{K_*} K_*K)$ is flat. But if you look here, you need an explicit formula, and this is K_*K and that map happens to be, I think I should write it out: you need to calculate $K_*\mathbb{CP}^\infty \to K_*\mathbb{CP}^\infty \otimes_{K_*} K_*K$. It's really lucky that we can work out what is this η_R. Then we map to $\hat{K}_* \otimes_{K_*\mathbb{CP}^\infty} (K_*\mathbb{CP}^\infty \otimes_{K_*} K_*K)$ This happens to be a flat map, it coincides with the amp $K_*\mathbb{CP}^\infty \to K_*K$. It's a localization, you invert v. The only thing you are missing in the image is v^{-1}. This is the theorem.

I go back and my functor is exact, as long as I make sure in the category of comodules over this Hopf algebroid I have enough projectives, the Tor is going to vanish. Relative projectives. Everything is specific because you're working in this special case.

There's nothing in the spectral sequence of twisted K-theory, unfortunately, and I'm done.

[Does anything carry over equivariantly?]

That's an interesting question.