# TOPOLOGY SEMINAR: BEHAVIOR OF QUILLEN (CO)HOMOLOGY WITH RESPECT TO ADJUNCTIONS 

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## [Next week's seminar will be on Tuesday.]

I'd like to thank the organizers, it's always a pleasure to come here. I'd like to talk about something that came up in my doctoral work at MIT. Let me give you a bit of motivation about the problem we're trying to answer. Let me give you the main reference, it's a preprint by the same name that is on my website or the ArXiv. If you go back to Quillen's original paper on cohomology for commutative algebras. He sets it up for arbitrary algebraic theories. But then he exhibits a spectral sequence whose $E_{2}$ term is somethnig involving Quillen homology of commutative algebras and which converges to something closely related to Quillen homology of associative algebras. My notation for Quillen homology will be $H Q$. It's shorter. The idea here is that if you want to compute $H Q$ it might be easier to do it in another category where it's easier to compute.

One common way of comparing categories is when you have an adjunction $L: \mathscr{C} \leftrightarrows \mathcal{D}: R$. The question today is what happens, what is the comparison on $H Q_{*}$ and $H Q^{*}$. The adjunction to have in mind is either $A l g_{R} \leftrightarrows C o m_{R}$. The other is $\Pi a l g \leftrightarrow \Pi a l g^{n}$, where the adjunctions are truncation and inclusion.

To tackle the problem from a general perspective we need to be on solid ground, so let me remind you of background where this makes sense.

Let me give some terminology. Say you have a category $\mathscr{C}$, a nice category. You should have a notion of module, which is an Abelian group object in the slice object, $A b(\mathscr{C} / X)$, so bimodules over a ring, or modules, a graded group with an action (in commutative rings) or group actions. You can forget the structure to $\mathscr{C} / X$, and this forget sometimes has an adjoint called Abelianization. Then $H Q$ should be derived Abelianization. Take simplicial objects, and then you will be able to do derived things. So Quillen homology, take a cofibrant replacement of the identity, so of $X$, call it $C_{\bullet} \rightarrow X$. Then the cotangent complex $\mathbb{L}_{x}=A b_{x}\left(C_{\bullet}\right)$. That's called the cotangent complex.
[The model structure is given by the underlying sets. Take homs from projective objects (which have lifts from epimorphisms)]

So $H Q_{*}(X)=\pi_{*} \mathbb{L}_{x}$ and cohomology will be homming into a simplicial module, $H Q^{*}(X, M)=$ $\operatorname{Hom}\left(A b_{x} C_{\bullet}, M\right)$, this is a cosimplicial Abelian group; take its cohomology using Dold-Kan, and that's Quillen cohomology. Here $M$ is an $X$-module. For example, I said that historically the most interesting example was commutative rings, so if $\mathscr{C}$ is $C o m_{R}$, if you take $A$ a commutative $R$-algebra, you have the multiplication $m$, and the kernel is $I_{A}$, and you can kill $I_{A} / I_{A}^{2}$, that's what we call the module of Kahler differentials $\Omega_{A / R}$ and that's $A b_{A} A$. Then $A b(B \rightarrow A)=$ $A \otimes_{B} \Omega_{B / R}$. The point is that you can compute in each category what the ingredients are.

What category do we need to work with Quillen cohomology. We're using the Quillen model structure so we need a structure on simplicial objects. If you look at Quillen's homotopical algebra, he gives sufficient conditions. For example, if a category is cocomplete and has a set of small projective generators. [Comment from Paul, that's not quite enough maybe.] We also want $A b_{x}$ to exist for any object, and we want the adjunction to be a Quillen pair. Another thing that's useful, if you want the Abelianization of any object over your ground object, you compute the Abelianization of the source over itself and push forward, that's the adjoint to pulling back. You want to have pushforwards. That means if you have $f: X \rightarrow Y$ you have a pullback functor $A b(\mathscr{C} / Y) \rightarrow A b(\mathscr{C} / X)$, and you want that $f^{*}$ to have a left adjoint $f_{!}^{A b}$. Think of it as tensoring with the target. Another very good but not necessary element, you want Beck modules $A b(\mathscr{C} / X)$ to be Abelian for any object. When I say it's not absolutely necessary, Quillen gives sufficient conditions, including this one, it's not necessary but it's convenient.

Then Quillen in his paper, defines an algebraic category as a cocomplete category with a set of small projective generators. By small, I mean hom from commutes with filtered colimits. Projective is retract of free essentially, and generators means that every object receives an epi from a coproduct of generators. For example, the free group on one element is a set of small projective generators. In Lawvere's work, [unintelligible], but there he assumes exactness which is a technical ingredient but actually important.

If $\mathscr{C}$ is algebraic, certainly you have the standard model structure, which is the reason for the definition, along with Abelianizations and pushforwards, but the Beck modules are not Abelian. The standard way to get that is to start with an exact category.

Tho upshot is that Abelian [sic] categories are a good place for Quillen cohomology. But we want the Abelianization functor to be part of a Quillen pair. When does that happen? When does an adjunction $\mathscr{C} \leftrightarrows \mathcal{D}$ prolong to a Quillen pair $s \mathscr{C} \leftrightarrows s \mathcal{D}$ ? If $\mathscr{C}$ and $\mathcal{D}$ have enough projectives and finite limits, then the following are equivalent:
(1) $L$ preserves projectives
(2) $R$ preserves regular epimorphisms.

If moreover there are additional conditions satisfied by $C c$ and $\mathcal{D}$, then these two conditions are equivalent to prolonging to a Quillen pair. So the proposition answeres precisely the question above.

It's probably in the literature.
Proposition 1. If you start with Algebraic categories, then you have a Quillet adjunction prolonged from the Abelianization functor. This is easy to check once you have the background. The upshot is that algebraic categories in Quillen's sense are a good setup for Quillen cohomology.

Now equipped with those tools we can look at the effect of adjunction. First, we want to put together the adjunctions we have. We have the adjunction in $\mathscr{C}$ and in $\mathcal{D}$ so we need to look at the effect on slice categories.

Proposition 2. The induced adjunction on slice categories, let's say we have a ground object c. Then we apply $L$ to $\mathscr{C} / c \rightarrow \mathcal{D} / \mathbb{L} c$. For the induced adjunction, you take $c$ to $R L c$ by the unit of the adjunction If you have the opposite, you use $L R d \rightarrow d$.

For Beck modules we need to be a little more careful. Remember these are Abelian group obects in the slice category. The induced adjunction on Beck modules, if we look at $A b(\mathscr{C} / a)$ then the right adjoint passes to Beck modules and the left adjoint is something you need to compute, you don't know it.

Let's assume $L$ passes to Beck modules. Then the left adjoint really is the original $L$. so $E \rightarrow X$ induces $L E \rightarrow L X$. The other picture, the right adjoint automatically passes to Beck modules. In this case the left adjoint passes to Beck modules if the counit is an isomorphism, that's way too strong. [unintelligible], and for today we'll only look at the first case.

Now we'll fit together the adjunctions in the "fundamental comparison diagram." Maybe I should just say it. [unintelligible].

So what's the comparison diagram when you piece everything together? If $\mathscr{C}$ and $\mathcal{D}$ are algebraic categories and $L, R$ is an adjunction that prolongs to a Quillen pair, and the left-adjoint passes to Beck modules (only for simplicity, this is not necessary). Then the following is four Quillen pairs:

[missed some] So you get $L\left(\mathbb{L}_{c}\right) \rightarrow \mathcal{L}_{L_{c}}$. Then $H Q_{*}, H Q^{*}$ (isos if this is a weak equivalence).
So say that you take associative and commutative algebras, then you have


Suppose $\operatorname{Tor}^{R}(A, A)=0$ for $i \geq 1$, then $H Q^{*}$ is Hochschild homology up to a shift in degree, $H H_{j+1}(A)=\pi_{j}\left(A \otimes_{\operatorname{Com}\left(C_{\bullet}\right)} \Omega_{C o m\left(C_{\bullet}\right) / R}\right)$ where $C_{\bullet} \rightarrow A$ is a cofibrant replacement. In particular, you get a comparison map $H H_{j+1}(A) \rightarrow H Q_{j}(A)$ and I believe that's an edge morphism in the spectral sequence that Quillen exhibited.

Let me conclude with the other example that I had in mind, if you have the Postnikov truncation of $\Pi$ algebras, then if $M$ is an $n$-truncated module over some $\Pi$-algebra $A$, you have a truncation isomorphism $H Q^{*}\left(P_{n} A, M\right) \stackrel{\cong}{\rightrightarrows} H Q^{*}(A, M)$ where the right side is computed in $\Pi$-algebras but the left side in $n$-truncated $\Pi$-algebras.

A $\Pi$-algebra is a graded group with homotopy group structure, you have Whitehead products, a $\pi_{1}$ action on $\pi_{n}$ and a precomposition by $\alpha: S^{k} \rightarrow S^{n}$, so these have relations from homotopy groups of spheres.
[Ezra makes a comment about $\Pi$-algebras versus higher structure.]

