# TOPOLOGY SEMINAR
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Thank you for letting me speak. The motivation of what I want to talk about is Morava $E$-theory. I'll tell you a theorem about moduli of elliptic curves. I feel like I'm well outside where I'm familiar. Let me start with something that is not either of those things. Let's look at $A$ an Abelian group, possibly infinite, and I'll define a cochain complex $\mathcal{K}^{\bullet}_{p^r}(A)$, where $p$ is a prime and $r \geq 0$.

**Definition 1.** *$\mathcal{K}^{\bullet}_{p^r}(A)$ will be built out of flags of subgroups. $\mathcal{K}^{q}_{p^r}(A)$ is ($k$ is a commutative ring)*

$$\prod_{G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_q \subset A} k$$

*where $|G_q| = p^r$. The boundary map $(\delta f)(G_1, \ldots, G_{q+1}) = \sum (-1)^k f(G_1, \ldots, \hat{G}_k, \ldots G_{q+1})$.*

**Proposition 1.** (1) *$H^q\mathcal{K}_{p^r}(A,k)$ is $0$ if $q \neq r$*

(2) *$H^r\mathcal{K}_{p^r}(A,k)$ is a free module of rank $np^{\frac{r(r-1)}{2}}$ where $n$ is the number of subgroups of $A$ isomorphic to $(\mathbb{Z}/p)^r$.*

For a froof, $\mathcal{K}_{p^r}(A,k)$ is a direct sum over $\tilde{C}^{\bullet-2}(P_G)$ where $P_G$ is the nerve of the poset of proper and nontrivial subgroups of $G$. If $G$ is not elementary Abelian then this is contractible.

Assume that $k$ is not characteristic $p$, algebraically closed, and let $E$ be an elliptic curve over $k$, and then let $A = E(k)$. Then the torsion in $p$ is $(\mathbb{Q}_p/\mathbb{Z}_p)^2$ so in this case, $H^1(\mathcal{K}_p(E(k),k) = k^{p+1}$, $H^2(\mathcal{K}_{p^2}(E(k),k) = k^p$ and all the others are 0. This is a combinatorial observation about the $p$-torsion.

I want to build a complex $\mathcal{K}^{\bullet}_{p^r}(E/S)$ which will be a cochain complex of $A$-modules, built from the moduli of chains or flags of subgroups. Then I'd like to prove a theorem. It will have the same form as the one before.

There's a moduli problem. What I know about this comes from Katz-Mazur. This is called $[N - Isog](E/S)$ which is the set of subgroup schemes $G \subset E$ over $S$ which are finite and flat over $S$ of rank $N$. If my curve was algebraically closed and the characteristic didn't divide $N$ then I would just be counting subgroups. They show that this moduli problem is representable. To state it in the form that I need it, it's representable by a stack that is finite and flat over the moduli stack of elliptic curves. If I fix $E/S$ where $S = Spec(A)$, then there's a ring $\mathcal{S}_N(E/S)$, a commutative $A$-algebra, and $E^1/Spec(\mathcal{S}_N(E/S))$ with a subgroup scheme. There exists data which are the universal example of an element in $[N - Isog]$ for any elliptic curve $\tilde{E}$ obtained from $E$ by pulling back along some map.

I'll have $A$-algebra maps from $\mathcal{S}_N(E/S)$ to $B$ which will count subgroups of [unintelligible].

This thing $\mathcal{S}_N(E/S)$ is locally free as an $A$ module, with rank equal to the number of subgroups of order $N$ in $S^1 \times S^1$. If I can do that then I can do the same thing for chains of subgroups. Similarly, there exists $\mathcal{S}_{N_1,\ldots,N_q}(E/S)$ which carries the universal example of $G_1 \subset G_2 \subset \cdots \subset E$ where $G_i/G_{i-1}$has rank $N_i$ over $S$.

**Definition 2.** *Keeping $S$ affine, $S = Spec\ A$, let $\mathcal{K}^\bullet_{p^r}(E/S)$ be $\prod \mathbb{S}_{p^{r_1},\ldots,p^{r_q}}(E/S)$ where $r_i \geq 1$ and $r = \sum r_i$. There are various ways that this could happen according to the order of the subgroups and I want them all.*

*There is a coboundary the way it was before.*

**Theorem 1.** (1) *This is a strong vanishing result. $H^j\mathcal{K}^\bullet_{p^r}(E/S) = 0$ if $j \neq r$.*
(2) *$H^r\mathcal{K}^\bullet_{p^r}(E/S)$ is finite and locally free over $A$. It's $0$ if $r \geq 3$.*
(3) *$H^0\mathcal{K}_{p^0} \cong A = \mathcal{S}_1(E/S)$*
(4) *$H^1\mathcal{K}_{p^1} \cong \mathcal{S}_p(E/S)$*
(5) *$H^2\mathcal{K}_{p^2} \cong \mathcal{S}_p(E/S)/\mathcal{S}_1(E/S)$.*

So for instace $\mathcal{K}_{p^0}$ just has $\mathcal{S}_1$ in degree 0, $\mathcal{K}_{p^1}$ has $\mathcal{S}_p$ in degree 1, and $\mathcal{K}_{p^2}$ is $0 \to \mathcal{S}_{p^2} \to \mathcal{S}_{p,p}$.

So, how do you prove such a thing? I already showed you one example. I took an elliptic curve over an algebraically closed field of characteristic not dividing $p$. The proofs reduce to the case of $E/Spec(k)$, where $k$ is an algebraically closed field.

You know that everybody in sight is finite and free. There are three cases. If the characteristic is not $p$ then $E[p^r]$ is étale. $k$ is algebraically closed, $E(p^r) = (\mathbb{Z}/p^r)^2$. This is the case I already did. The other two cases have characteristic $p$, where $E$ is either an ordinary or a supersingular curve.

You can prove these cases by a calculation. That will be the proof. The complex itself doesn't depend on the elliptic curve. This really only depends on the $p$-divisible group (over a field). There are many nonisomorphic curves but if I'm working over an algebraically closed group, I only need to look at [unintelligible]. The supersingular, you look at [unintelligible]. You don't have to look at the ordinary case even. I'm, it turns out that the vanishing condition is a local condition. If I prove it at a point, then [unintelligible]. If I prove it for a supersingular curve, then I know it in a neighborhood, knowing it for an ordinary curve, and then I know it for any ordinary curve.

This is the case that is relevant to Morava $E$-theory, but maybe I won't develop that yet. I've been talking about proving this, but I've mostly been showing the vanishing. Once you know the freedom, that's just by dimension counting. Let me show you about $H^2$. I want to look at $H^2$ of the complex, I have $\mathcal{S}_{p^2}(E/S) \to \mathcal{S}_{p,p}(E/S)$. I've emphasized subgroups. Any finite subgroup locally free over the base induces an isogeny. $Ker(f) = G$. I'm going to think of $g$ as carrying the universal isogeny of order $p^2$. If I have a pair of subgroups arranged in a flag, I can think of that as a pair of composable morphisms, each of degree $p$. I want to describe a morphism. The morphism on moduli problems goes the other way, it'll take this flag to the the composite. So $H^2$ is also given as a quotient of the map $\mathcal{S}_1(E/S) \to \mathcal{S}_p(E/S)$ where I just forget. An isogeny of degree $p$ will give $f$ and then the dual $\hat{f}$. On the other side, $E$ the elliptic curve associates $[p]$. It's a pullback square of rings. It's a pullback square of modules. I can form the quotient andI should have an isomorphism. You prove this, you know from the part you proved, you know the cohomology has to be locally free of a certain rank. You just have to show that it's surjective.

Both of the vertical maps are surjective because there are sections. Then you know by rank that this is a pullback square. That's how you prove that.

Now let's think about the supersingular curve. I won't do it directly, I'll prove something about its universal deformation. I'm going to take a supersingular curve $E_0/k$ where $k$ is characteristic $p$ and $k = \mathbb{F}p^2$. Suppose, I'm going to take algebras which are Artin local $k$-algebras, look at deformations of this curve to $k$-algebras. I'll associate to this a category $Def_{E_0}(A)$. The objects of this category will be triplets $(E, \psi, \alpha)$, where $E$ is an elliptic curve over *Spec A*, $\psi$ is a map $\mathbb{F}_{p^2} \to A/\mathfrak{m}$, and $\alpha : E \otimes A/\mathfrak{m} \xrightarrow{\sim} \psi^* E_0$.

Given two objects $E_i, \psi_i, \alpha_i$, a deformation of $F^r$ is an isogeny $f : E_1 \to E_2$ such that

(1) $\psi_2 = \psi^1 \circ \sigma^r$ where $\sigma$ is the $p$-power map, and
(2) the diagram commutes:

$$\begin{tikzcd} E_1 \otimes A/\mathfrak{m} \arrow[r, "f\otimes N_{\mathfrak{m}}"] \arrow[d, "\alpha_1"'] & E_2 \otimes A/\mathfrak{m} \arrow[d, "\alpha_2"] \\ \psi_1 E_0 \arrow[r, "F^r"'] & \psi_2^* E_0 \end{tikzcd}$$

I want to describe the structure of this category. The first thing I want to say is that there is a universal deformation $E_{univ}, \psi_{univ}, \alpha_{univ}$ over $\mathbb{F}_{p^2}[[x]]$ which is universal for deformations of $E_0$ up to isomorphism. Isomorphism of deformations is a morphism of the category which is an isomorphism. So this should be a deformation of $F^0$ which is the identity.

The solution to this is the same as the solution to [unintelligible]. This is really that theory again. There's a universal deformation. The ring $\mathbb{F}_{p^2}[[x]]$ classifies deformations up to isomorphism. I also like to classify morphisms.

**Proposition 2.** *Let $E_0/\mathbb{F}_{p^2}$ be a "standard" supersingular curve. Then there is a universal example of a deformation of $F^r$ which lives over $\mathscr{F}_{p^r}[[x_1, x_2]]/F_{p^r}(x_1, x_2)$ where $F_{p^r}(x_1, x_2) = \prod_{i+j=r}(x_1^{p_i} - x_2^{p_j})$.*

I've described an abstract ring that classifies these deformations. There are two maps $\mathbb{F}_{p^2}[[x]]$ one sending to $x_1$ and the other to $x_2$. Writing a composition depends on these.

**Remark 1.** *If you take $F_{p^r}(x, y)$ and $F_{p^{r'}}(y, z)$, then the ideal generated here contains $Fp^{r+r'}(x, z)$. This looks highly improbable at first glance but then it turns out to be provable without using all of this.*

I didn't say what standard was. $E_0/\mathbb{F}_{p^2}$ is standard if $F^2 = [-p]$. Every supersingular curve is isomorphic to one of these. I learned this on mathoverflow. You want to know (to use Katz-Mazur) that the dual of Frobenius is a deformation of Frobenius. You need to understand that.

What does this have to do with my story? I'm interested in $\mathcal{K}_{p^r}(E_0/k)$. Let me show vanishing of cohomology for $E^{univ}/k[[x]]$. You can write down everything about the complex explicitly. The rings $\mathcal{S}_{p^{r_1}\dots p^{r_q}} \cong k[[x_0, \dots x_q]]/(\dots, F_{p^{r_i}}(x_{i-1}, x_i), \dots)$

Let $L = Hom_{k[[x]]}\mathcal{K}^{\bullet}_{p^r}(E^{univ}/k[[x]]), k[[x]]$. The homoloogy is concentrated in degree $r$. I can give a concrete description of the dual complex which is the bar complex of a certain algebra.

**Proposition 3.**
$$\bigoplus_r L_{p^r} \approx \bar{\mathcal{B}}(k[[x]], \Gamma, k[[x]])$$
*where* $\Gamma$ *is generated over* $k[[x]]$ *by* $P_0 x = -x^{p+1}P_p$, $P_1 x = P_0 + xP_p$, $P_{i,x} = P_{i-1}$ *for* $1 < i < p$ *and* $P_p(X) = P_{p-1} + x^p P_p$.

*[missed something.]*

*There's a second set of relations, the real ones,* $P_iP_0 + xP_iP_1 + \cdots + x^pP_iP_p$ *for* $i = 1, \ldots, p$.

If you think about this complex, the algebra structure encodes composition of isogenies. The relations are quadratic, at least modulo the couple. To check that this is zero, check that this is Koszul, just as Stewart does in the original paper.

What does this have to do with Morava $E$-theory? Strickland showed that the ring spectrum of a generalized cohomology theory has an algebra $\mathcal{P}$ of power operations. which is associated to the moduli of finite subgroups of a formal group. This is just in analogy with the situation that I've described. This calculation gives or describes $\mathcal{P} \otimes \mathbb{Z}/p = \Gamma$. The real calculation you'd like to understand is not characteristic $p$ This calculation [unintelligible]for $\hat{E}_0$ (standard).