GEOMETRY/PHYSICS SEMINAR
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1. Pretalk: Intro to Khovanov homology

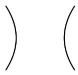
Let me begin at the beginning. Alexander discovered, long ago, how to make a polynomial out of knots and links. That would be a good place to start. He figured out how to write a polynomial in terms of a knot diagram. It was a bit mysterious about what it meant. He gave a combinatorial definition which uses a result of Reidemeister which we will use as well. He wrote the book "Knoten Theorie." He stated and proved that the three diagramatic moves RI, RII, RIII, capture the data of knots in three space. You have a larger diagram, and you are allowed to apply these moves locally on the larger diagram. The diagram is a 4-regular graph in the plane with extra structure of a crossing that goes one way or the other way in relation to the vertex. So this is a formalization into graph theory of some structures related to the topology. Then you're allowed to find and apply one of these to any graph locally.

If you know that any isotopy can be realized by a sequence of these moves. If there were an infinite number of types of loop removal you'd be in trouble.

He proves it, by the way, By the way, can we unknot this picture? Here's a 2-move, a 2-move, here's a 3-move. That isn't the most efficient use. It all goes away, here's an unknot. Reidemeister's theorem is that two knots are equivalent if you can move one to the other by a sequence of these three moves. Maybe thing there's a 0 move, that you can do homeomorphisms of an uncrossed part of the graph.

There's a simple algorithm to take a diagram into an embedding. For example, a famous and useful combinatorial data is to label each crossing, so you get $\overline{1}\underline{2}\overline{3}\underline{1}\overline{2}\underline{3}$. You can ask a computer to work with something like this. The diagrams are an intermediate stage. That's Reidemeister. Alexander used these moves and explained how to write down a certain determinant and that was the polynomial. People started to understand that this was related to the fundamental group of the complement. Seifert figured out how to think of it in terms of surfaces bounded by the knot. It is a fact thatif you give me a knot then you can find a surface whose boundary is the knot. You can checkerboard color the diagram. One region and another have adjacent colors. By the Jordan curve theorem you can always color in this fashion. This might not be orientable. One of my favorite facts is that if you take a M obius band that is three-twisted instead of 1-twisted and cut it in half down the middle, you'll get a trefoil. You can easily prove that every knot has a surface that's orientable with boundary the knot. You could derive the Alexander polynomial by thinking about surfaces in three-space. This could be expressed in terms of more invariant notions than diagrams. This interfaces with algebraic topology, write down a definition that is invariant and then work to compute it.

We're not there yet with the Jones polynomial. These invariants will not be tied to the Reidemeister moves forever. For the moment, they are. I'll jump from the 1920s to the 1980s and show you the bracket model for the Jones polynomial.

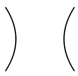
I have a knot diagram. I won't orient it yet. Given the diagram  it's possible to replace it with  or . Here's an example with the trefoil: [picture]. If we color the four sides, labelling the left of the undercrossing A and the right B, then we call these the A and B smoothing, respectively.

We'll define a polynomial $\langle K\rangle \in \mathbb{Z}[A, B, d]$ by the rule that

$\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$

along with the rule that a disjoint loop adds a factor of d and that the empty diagram has polynomial 1. So if we do this to the Hopf link we get this calculation: [pictures]

The resulting polynomial is $A^2d^2 + ABd + BAd + B^2d^2$. There are two things I want to do. This won't be invariant under Reidemeister moves. There are four configurations coming out of the smoothing process. If you label the smoothings, you see that the variables are the variables coming out of these configurations. I can read off the contributions to the polynomial. So another way of putting how the polynomial gets evaluated is as a sum over these configurations S that I call states, I tend to write $\langle K\rangle = \sum_S \langle K|S\rangle d^{||S||}$ where $\langle K|S\rangle$ is a product of A and B according to the smoothing chosen and $||S||$ is the number of loops. You realize we could have taken this as the definition and then this is clearly well-defined. You might worry that if I did this in a different order I would get something else.

That's the definition of the raw polynomial. Going back to that is what we'll do in a few minutes. But now I'll ask how this behaves under the Reidemeister moves. You might choose RIII first but we'll choose RII first. The answer is, on the complicated side, $AB\langle$  $\rangle + (ABd + A^2 + B^2)\langle$  $\rangle$. If you take $B = A^{-1}$ and $d = -A^2 - A^{-2}$ then $\langle K\rangle$ will be invariant under RII (and RIII). I only checked two. Three is easy to check so we'll do it. What happens with III. If you do the calculation, [pictures]. What happens under RI? We can check. We find out that we get a factor of $Ad + A^{-1}$ which is $-A^3$. If you have a little curl, it multiplies by $-A^3$, and the reverse curl multiplies by $-A^{-3}$. You can then regard this as an invariant of framed links or normalize it, saying $f_K = (A^3)^{-wr(K)}\langle K\rangle/\langle\bigcirc\rangle$ where the writhe $wr(K)$ is the sum of $\pm$ over so-called positive or negative crossings. Compensating in this way, the term I add is invariant under RII and RIII as well, so this is invariant under all of these. This is the Jones polynomial with a change of variables $V_K(t) = f_k(t^{-\frac{1}{4}})$. Jones discovered this while looking for a way to represent the braid group into the Temperly-Lieb algebra. He thought there ought to be a representation, and continued talkind to knot theorists, including Joan Birman. He did this by taking a trace of his representation. It's related to what I'm showing you. I could show you how this is related to taking a trace, that's another story. There are a lot of stories that come out of this interpreting it in different ways. You might think of putting matrices on the lines, standing this up. If you were careful you'd separate the critical points. Then you have a Morse diagram that you would divide into bits and pieces. You should have mappings of modules to modules or modules to the base ring or the base ring to a module. The scalar that comes from this should be a knot invariant. This fits into that story.

Khovanov was trying to generalize this into higher dimensions. Then you try to uplevel the categories involved. So you look for maps $V \otimes V \to V \otimes V$ or $\neg \to V \otimes V$. Here you have something simple like modules or a number, and maps, you could try to uplevel the flavor by making the level categories and then the upleveling functors.

You might be looking at a topological problem up one dimension. I put the surface in fourspace into a nice position. If I slice with a hyperplane I have a link. I see knots changing into other knots. As I go from one step to the next, I'm going to see two circles go through a saddle and get to a single circle. There are a few basic cases that you could see. Then what happens is, at the base where you hoped you had a matirix, you had the thing calculating knot invariants. Above you have something else, and a morphism of a morphism. A functor from some category to another. When he was working on this problem, he was thinking about how to categorify these, and certain ideas that naturally come up when you're looking at this, he found a creative way to combine these ideas. You're thinking, well, I shoud think carefully about saddle points. They do something like this. But saddle points are morphisms that take you from a smoothing one way to a smoothing the other way. There should be mappings between smoothings that go one way and smoothings that go the other way. You could look for mappings into algebras that do the right thing. These could be building a chain complex and you could compute a homology. I could try to reconstruct by thinking about the state diagram for a knot. This is something like that. You have, say, an A-state and the others are obtained by resmoothing. The other states come from this one by turning some As into Bs. Then if you look at this a little more abstractly, I have, I can turn any of these into a B, and then I could change two of them, or I could change all three.

This is a description at an abstract level of the 8 states of the treofoil knot. The states are in a hierarchy. Abstractly, the structure is that of a cube

$$\begin{tikzcd}
& AAA \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
BAA \arrow[d, dash] \arrow[dr, dash] & ABA \arrow[dl, dash] \arrow[dr, dash] & AAB \arrow[dl, dash] \arrow[d, dash] \\
BBA \arrow[dr, dash] & BAB \arrow[d, dash] & ABB \arrow[dl, dash] \\
& BBB &
\end{tikzcd}$$

Here you're not getting anything interesting yet, but if you map $SC(K)$ into the module category then you will get something interesting. You need to take a loop to what, maybe, a module. Then if that's the case I have very little choice but to take two loops to the tensor product $V \otimes V$. Then everything is catelogued by the combinatorial structure, which gives boundary maps. So what's the structure of the boundary maps? The ansatz says that the boundary of x should be the sum of local boundaries. Each one of those sites has the possibility of a local boundary. There are a whole slew of module maps. Surely there should be. Here in this picture there's boundary number one, two, and three. Here there are two boundary maps, and here one. Once it's become B it can't be done again. Working modulo 2, you simply put signs in later. What do we need? If $\delta x = \sum \delta_i x$ and we want $\delta^2 x = 0$, this will only be true if $\delta_i\delta_j = \delta_j\delta_i$. What about the boundary maps. Our local boundary for this picture would would a boundary $V \otimes V \to V$. We might also do this: $V \to V \otimes V$. So going through a saddle point means going from two to one or two one, m or Δ. So m is like a multiplication and Δ is a comultiplication. So to have a boundary mapping, we'll need a product and a coproduct. The axioms are up in the air for the moment. So, this is the key, I need to see the compatibility, so let's look. If we have three things like this, before I do simple or complex, I want to introduce a way of drawing pictures of these boundary maps coincident with our philosophy. Draw going through a saddle as an m and in the opposite way as the Δ. So I can think of each individual boundary map and bit of one of these as a cobordism of 1-manifolds. So then if I wanted to do 1 and then 2, I'd multiply here first, then here. What this is telling you is doing boundaries in this order is the same as multiplying along $m(m \otimes 1)$. If I did it in the other order, if I did it with elements, I'd get $(ab)c$ and $a(bc)$. The compatibility condition here is associativity. If we turn it upside down the compatibility is coassociativity. What you'll find is that the compatibility conditions are given by the surfaces, homeomorphisms of the intervening surfaces. The right algebra is $k[x]/(x^2)$ with $\Delta(x) = x \otimes x$, and $\Delta(1) = 1 \otimes x + x \otimes 1$. That's the complete description of the structure of the boundary mapping. Miracle of miracles, the homology of the complex, with appropriate understanding of the shifts, is an invariant of the knot and recovers the bracket.

2. Categorifications of the arrow polynomial

[I do not take full notes at slide talks]

I will show you how the arrow polynomial is a kind of generalization of the Jones polynomial. Here are references. I'll use combinatorics, Reidemeister moves. If you represent knots by diagrams then two representations are isotopic if and only you can get from one to the other by these moves plus planar isotopy of the graph. If you try to do these in combinatorial topology, it comes down to this. For Khovanov you'd like your definition to be more general. That puts the subject in the old days of combinatorial topology, but it will take a lot of reformulations. The Reidemeister moves are nice, and the invariant we'll talk about first comes from this model of the Jones polynomial, where you make an invariant by taking a diagram and splitting each crossing, and adding up over every possible way of smoothing. If you are looking at a single diagram, you take the two associated diagrams where the smoothing has been made, and multiplying by various factors. If you want this to be invariant, you need this for well-definition. This gives the original Joen's polynomial by $f_k(t^{-\frac{1}{4}}) = (-A^3)^{-wr(K)}\langle K\rangle/\langle \bigcirc \rangle$. The writhe is the sum over the crossings of ± 1. That means if you were calculating the trefoil knot the writhe would be $+3$. That's the relationship between this kind of modelling and the Jones polynomial.

Reversing the orientation on a knot you get the same result. For links you might get a different result. There are no isotopies between these two version of the Hopf link. That's the bracket model. I'm going to discuss the bracket model a little carefully by rewriting it a little bit. Multiply the entire formula by $A^{-c(K)}$. This is just bookkeeping. Any time you had an A in the expansion it'll get removed. You'll get A^{-2}. I can set things up so that I have q. Then the value of the loop is $q + q^{-1}$. You think that the value of the loop should be two loops, one for q and the other for the q^{-1}. Instead of having a single loop, have a $+$ loop and a $-$ loop. Then every evaluation is monomial. I guess I should say one other thing that the slides didn't say. If I were calculating the bracket for the Hopf link here, you see that there are four configurations of the diagram that are relevant to me for doing this calculation. One of them is like this. The other three are obtained from this by replacing As by Bs and resmoothing. I'm putting a B in for A^{-1}. [Picture]. You can define the bracket to be the sum over states of a combinatorial formula (see above). I call this the state sum for the bracket. It has the recursive formula but you can write it as a sum over states. Then you would have, if you did it that way, if $d = q + q^{-1}$. When you do it this way, you have to raise to various powers. If we did it the other way, each one of these gives rise to more, and these would be the enhanced states $\mathcal{ES}$. The binomial theorem disappears into all of the possibilities for labelling. Suppose you were looking at $(q + q^{-1})^2$. Then each is labelled $+$ or $-$, you get four terms, which will be the four terms coming out of squaring this. Going to the enhanced states the powers disappear but you get lots more states. It makes sense to call -1 an X and $+1$ a 1. Here's an example of that kind of contribution.

The enhanced state sum formula is $\langle K \rangle = \sum_s q^{j(s)}(-1)^{i(s)}$.

If I were to rewrite it in counting it of $(-1)^i q^j \dim C^{ij}$ where C^{ij} is a module generated by enhanced states with i and j as their gradings. This is an Euler characteristic of something, if it were a chain complex. This actually fills in for Khovanov homology. When you define the boundary in this gadget, the basic point about the boundary is that the boundary is composed of pieces and on a boundary piece it corresponds to a resmoothing. I haven't told you what happens to the algebra, but the form of the states, this is a boundary piece. It gets resmoothed like this. When you apply boundary, you go C^i to C^{i+1}. The j remains fixed. This is the way it looks when we get the boundary fixed which we will in a moment. You start with entirely As. I start with all As, smooth once to get one B, and increase i by one each time. The j remains the same and when one speaks of Khovanov homology, one cuts out a piece that just has j. When one says it's a graded Euler characteristic, it's the sum of q^j times an actual Euler characteristic. The definition comes right out of this version of the state sum.

Here's the bare bones of what this complex will look like [Picture.]

Khovanov's idea was to take this bare structure, and think of it as a category and take this to modules instead of just those bare objects. Each one of these should become a module and each of these should be a map of modules. I'm avoiding signs by thinking mod two. If you take it to modules, then the circle should go to V. Here I'm talking about the states unenhanced.

So I'll tell you, $V = \daleth[x]/(x^2)$ where $\daleth = \mathbb{Z}_2$ (but that's irrelevant). Now I'm saying that these enhanced states correspond to the generators of V. This is part of my multiple think in terms of the diagram.