Deformation Theory and Operads

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I don't know whether these definitions are on wikipedia.

Definition 1 *An associative algebra A is rigid if it has no nontrivial formal one-parameter deformations. A is infinitessimally rigid if it has no nontrivial deformations over $k[t]/t^2$.*

A couple of quick remarks. A is infinitessimally rigid if and only if the Hochschild one-cohomology of A vanishes. If A is infinitessimally rigid then it is rigid.

Gabriel asked, if A is rigid, then is it infinitessimally rigid? Is there a good example?

Last time we checked, if $\mu = \cdot + \mu_1 t + \cdots$ where μ_i is a formal deformation of $(A, \cdot)$, then μ is associative implied that $\delta(\mu_1) = 0$, where δ is the Hochschild differential. You also get an equation $\delta(\mu_2) = \mu_1 \circ \mu_1$. Now I'll give an example of $(A, \cdot)$ and μ_1 so that $\delta\mu_1 = 0$ but $\mu_1 \circ \mu_1$ is not a coboundary, so that μ cannot be extended to an associative product.

I'm not going to do every detail, but here's my example. Let $A = k[x, y, z]/x^2, y^2, z^2, xy - z$. Consider μ_t' to be the product on $A[[t]] = \mu_t' = \cdot + \mu_1' \cdot t$, where $\mu_1'(y, x) = z$, and zero for all other products. This defines an actual deformation. This is infinitessimal, but it is actually a deformation over $k[[t]]$. There's another deformation, μ_t'' defined by $\mu_t'' = \cdot + \mu_1'' \cdot t$ where $\mu_1''(x, x) = y$.

This defines a deformation of A by deforming $x^2 = 0$ to $x^2 = ty$. Check that $\mu_1' + \mu_1'' : A \otimes A \to A$ is Hochschild closed, not an associative product, and cannot be extended to one.

Now consider $A = k[x, y, z]/xy, xz, yz$, which is the coordinate ring for the x, y, and z axes. Try to find an obstructed infinitessimal deformation.

A very simple example, for $A = k[x]$. Then μ_1 defined by $\mu_1(x^i, x^j) = ijx^{i+j}$. Then $\cdot + \mu_1 t$ is associative mod t^2. I can add a μ_2 term $\frac{i^2j^2}{2}x^{i+j}$. Then $\cdot + \mu_1 + \mu_2 t^2$ is associative mod t^3

As an exercise, give formulae for μ_i so that the whole sum is associative.

That's the end of the examples. Now let's look at the complex a little more closely.

Let me be precise. The Hochschild cochain complex is $C^\cdot(A, A)$. This notation is better and easier to stick to. The second entry is the same as the first entry, but it's not reduntant, if M is an A module then I can define $C^{\cdot}(A, M)$. The differential goes up. This will be neither contravariant nor covariant.

I would prefer to think of A first as being a naked graded vector space. It will be a sequence of modules. Then define $C^{\cdot}(A, A)$ by $Hom(A[1])$. FOr each n, this is a graded vector space. We define $C_k(A, A)$ to be the degree k maps.

In particular, if A were concentrated in degree zero, then elements of C^k are isomorphic to maps $A^{\otimes n+1}$.

I'm calling this, well, I want to define a circle product

$$f \circ g(a_1, \ldots a_{r+s+1})$$

I want to do this with signs. So when you pass g over a_1 you get a sign of $|g|(|a_1| + 1)$ (you are dealing with shifted things).

To make this nicer, consider $Hom(T(A[1]), A[1])$. The tensor algebra is the cofree coalgebra. You can lift any map like this to be a coderivation. Then you can bracket two coderivations, and that's determined by another such linear map. The result of that process will be the commutator of that circle product.

To a graded vector space, we have assigned $C^{\cdot}(A, A)$ and $\circ : C^i \otimes C^j \to C^{i+j}$. This is not associative but it is pre-Lie.

A graded Lie algebra is a pair $(V, [\,,\,])$ where $[\,,\,]$ satisfies $[f, g] = -(-1)^{|f||g|}[g, f]$ (skew symmetry) and $[[f, g], h] + (-1)^{|h|(|f|+|g|)}[[h, f], g] + (-1)^{|f|(|g|+|h|)}[[g, h], f]$ (graded Jacobi). This gives a preLie structure, so that the commutator is Lie.

Let's look at why the this is not associative. You're summing over inputs, so you get terms that don't cancel where one thing is under two things.

So here's a fact. What's nice about a graded Lie algebra $(V, [\,,\,])$. If x is even, so if the ground field has $2 \neq 0$ then $[x, x] = 0$. If x is odd then $[x, x]$ may not be zero. If it is zero, then the operator $ad\ x$, also, $\delta : V \to V$ defined by $y \mapsto [x, y]$ is a square zero operator. $\delta^2 y = [x, [x, y]]$, which equals $\pm\frac{1}{2}[y, [x, x]]$ So if $[x, x] = 0$ then $\delta^2 = 0$. If A is a graded vector space equipped with a map μ with is degree 1 and satisfies $\mu \circ \mu = 0$ (same as $[\mu, \mu] = 0$ outside characteristic two), then $C^{\cdot}(A, A), \delta$ is a cochain complex.

The only piece of information I added was assuming μ was degree one. Notice if μ has degree one, then bracketing with it will be a total degree one map from V to V.

An element μ which satisfies $[\mu, \mu] = 0$ is called an A_∞ structure on A and the pair (A, μ) is called an A_∞ algebra.

Suppose $\mu : A \to A$. For this to be degree one, it will still be degree one, regardless of the shift. $[\mu, \mu] = 0$ if and only if $\mu^2 = 0$ so (A, μ) is a complex.

If μ is a two to one map, then $[\mu, \mu] = 0$ is the same as μ being an associative algebra, so these are examples of A_∞ algebras.

Suppose $\mu = \mu_1 + \mu_2$. Then unpacking things you find out $[\mu_1, \mu_1] = 0$, $[\mu_2, \mu_2] = 0$, and $[\mu_2, \mu_1] = 0$ so that this is a differential graded associative algebra.