Deformation Theory and Operads

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July 9, 2012

I ended last time with two homework problems. I defined A_t as $\mathbb{R}[x]/x^3 - tx$. The first question is, is A_1 isomorphic to A_2. The other question is whether A_0 is isomorphic to A_t. Maybe I won't spoil the answer for anyone. It's a slightly interesting problem, actually. Let me make a definition. Let $(A, \cdot)$ be an associative algebra over a field $\mathbf{k}$. Then a formal one-parameter deformation of $(A, \cdot)$ is an associative algebra structure $\mu : A[[t]] \otimes_{\mathbf{k}[[t]]} A[[t]] \to A[[t]]$ such that the map such that the map $A[[t]] \to A$ (evaluation at zero) is a map of algebras over $\mathbf{k}$. You could assemble this from what I said last time except that I didn't say $A[[t]]$. This is actually $A\hat{\otimes}k[[t]]$. Generally you want to deal with the completed tensor product. Just to point out the difference: If I do $A \otimes k[[t]]$, this would be finite sums of $a_i \otimes p_i(t)$. If I take $A[[t]]$, this is infinite sums, power series in t with coefficients in a. The first one only allows finitely many distinct things from A. If $\mathbf{k}$ is topological, you can ask that this converge but in general you won't ask. A formal one parameter deformation of $(A, \cdot)$ is determined by a collection $\{\mu_i\}_{i=1}^{\infty}$ of bilinear maps $A \times A \to A$ as follows, $\mu(x, y) = x \cdot y + \mu_1(x, y)t + \mu_2(x, y)t^2 + \cdots$ I wanted to frame the second homework question in this language. If μ is a one-parameter formal deformation of $(A, \cdot)$ then μ is associative is equivalent to an infinite collection of relations among the $\{\mu_i\}$. In particular, μ_1, well, $\mu(x, \mu(y, z)) - \mu(\mu(x, y), z) = 0$. This is $x \cdot \mu(y, z) + t\mu_1(x, \mu(y, z) + \cdots - (\mu(x, y) \cdot z + t\mu_1(\mu(x, y), z)$

I can expand the other μ and eventually get some equations that I can write down. If these will be zero over $A[[t]]$ then each coefficient of t^n is zero. The constant coefficient is 0 so $(xy)z - x(yz) = 0$ and $x\mu_1(y, z) + \mu_1(x, yz) - \mu_1(x, y)z - \mu_1(xy, z)$ I wanted to isolate the t^1 condition. Let me write this in an easier way to remember.

That is, $x\mu_1(y, z) - \mu_1(xy, z) + \mu_1(x, yz) - \mu_1(x, y) \cdot z = 0$.

Let $(A, \cdot)$ be a $\mathbf{k}$ algebra. Two deformations μ and μ' of $(A, \cdot)$ over R are equivalent if and only if there exists an isomorphism of R-algebras $\varphi : A \otimes R \to A \otimes R$ so that the diagram commutes:

$$\begin{tikzcd}
A \otimes R, \mu \arrow[rr] \arrow[dr] & & A \otimes R, \mu' \arrow[dl] \\
& (A, \cdot) &
\end{tikzcd}$$

In the case of two formal one parameter deformations μ and μ' they are equivalent if and only if there exists a sequence of linear maps $\varphi_1, \cdots$ so that for all x and y, $\mu(x,y) = \varphi_t^{-1}\mu(\varphi_t(x), \varphi_t(y))$ where $\varphi_t(x) = x + t\varphi_1(x) + \cdots$

Given an associative algebra $(A, \cdot)$ over a field $\mathbf{k}$, one always has the very trivial defined by μ_t is trivial if and only if it is equivalent to the veryt trivial one.

[[unintelligible]]

Now let me erase everything on the board. I want to define the Hochschild complex.

Everyone can make up their own example of a trivial deformation. Pick any old maps.

Definition 1 *$C^n(A) := Hom_{\mathbf{k}}(\underbrace{A \otimes \cdots \otimes A}_{n+1}, A)$. Then let $C := \bigoplus n \geq 0 C^n$. This is a graded vector space. I want to define a differential $C^{n-1} \to C^n$ by $(\delta f)(a_1, \ldots, a_{n+1}) := a_1 f(a_2, \ldots, a_{n+1}) - f(a_1 a_2, a_3, \ldots, a_{n+1}) + \cdots + (-1)^n f(a_1, \ldots, a_n a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}$.*

You might ask about the grading. Let me finish the construction. Fact, exercise, $\delta^2 = 0$.

So if $f : A \otimes A \to A$ then $\delta f(x,y,z) = xf(y,z) - f(xy,z) + f(x,yz) - f(x,y)z$. The formula that was there before shows up, and we see that if μ is a formal deformation of $\cdot$ then $\delta\mu_1 = 0$. How shall I summarize this? This leads in Kodaira-Spencer terms to the "question of existence": Given $(A, \cdot)$ and a Hochschild 1-cocycle μ_1, does there exist a formal 1 parameter deformation μ of $(A, \cdot)$ so that $\mu(x,y) = x \cdot y + t\mu_1(x,y) + \cdots$

Let me make this an exercise. Show that if μ is a trivial 1-parameter formal deformation then μ_1 is a Hochschild coboundary. That is, there exists a $\phi_1 : A \to A$ so that $\delta\phi_1 = \mu_1$.

This is a special case. If two deformations are equivalent, then, if μ and μ' are equivalent then $\mu_1 - \mu'_1$ is a Hochschild coboundary. So the infinitessimal deformations, modulo equivalence, are the Hochschild 1-cohomology.

The question of existence is typically hard.

I can make the following definition

Definition 2 *Given an associative algebra $(A, \cdot)$ and a formal one parameter deformation μ_t, we define $\frac{d\mu_t}{dt}|_{t=0}$ to be $[\mu_1] \in H(C^1, \delta)$. The idea is that Hochschild 1-cohomology classes are infinitessimal deformations of $(A, \cdot)$.*

Let's go to graded algebras. Well, Gerstenhaber's circle product. Given $f \in C^r, g \in C^s$, define $f \circ g \in C^{r+s}$ by $f \circ g(a_1, a_2, \ldots, r + s + 1) = f(g(a_1, \ldots, s+1), a_{s+2}, \ldots) + (-1)^s f(a_1, g(a_2, \ldots, a_{s+2}), \ldots, a_{r+s+1}) + (-1)^{2s} \cdots$ So these signs will either all be plus or will alternate, depending on the sign of s.

I like to think of this pictorially. [Pictures] This is f with some inputs and an output. I let g

act on some number of inputs. So I evaluate g inside of f like this, plus or minus, this, plus or minus this, and so on.

Here's a fact. The associative product $\cdot$ on A is an element of $C^1(A)$ since it's a map from $A \otimes A \to A$. I'm already saying it's associative. I want to say that $\cdot$ is associative is equivalent to saying $\cdot \circ \cdot = 0$.

The condition that $\delta^2 = 0$, well, you have to use the associativity of $\cdot$ for that. The differential δ can also be defined by $\delta(f) = \cdot \circ f - (-1)^{|f|} f \circ \cdot$.

Let's see that this is the same. $\cdot \circ f(a_1, \ldots, a_{n+1})$, I evaluate f, so it's $f(a_1, \ldots, a_n)$

I skipped one part of trying to get the signs and I see I paid for it. Elements that $\circ$ with themselves and give zero, those are interesting. If you're in C^1, and do that, it's associative. The interesting 1 to 1 things are differentials, but those are only in graded algebras. The first thing I'd like to say, is how can I tell that $C^n = Hom(A^{\otimes n+1}, A)$ should be degree n. Let me give a practical reason. Given a graded vector space V, define the shift $V[k]$ for an integer k to be the graded vector space where $(V[k])^j = V^{k+j}$. If I shift by one, for example, then $V[1]^j = V^{j+1}$. If V were in degree zero, then $V[1]$, well, everything will be concentrated in degree -1.

If φ is in $Hom(V, V)$ has degree j, then $\varphi : V^i \to V^{i+j}$. Or a map $\varphi : V^i \to V^k$ has degree in the homomorphism complex (forget complex, this is a vector space), $k - i$. So if A is a vector space that's not graded, you can view it as graded and concentrated in degree zero. Then any map $\varphi : A^{\otimes n+1} \to A$ can be viewed as a map $\varphi : (A[1]^{\otimes n+1} \to A[1])$ of degree n.

The formula for the circle product will be a little different for a graded vector space, but the Hochschild complex is robust enough to give you things where square zero elements are interesting.