Deformation Theory and Operads

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July 9, 2012

I want to do at least one more example. The other example I did was

1. $A = TV$.

2. Let $A = SV = TV/(v_iv_j = v_jv_i)$ be the symmetric algebra on V over k. Denote by $\wedge^{\cdot} V$ the exterior algebra, $TV/(v_iv_j = -v_jv_i)$. Denote $u \odot v$ the product in SV and $u \wedge v$ the product in $\wedge V$, and $u \in SV$ is $u_1 \odot u_2 \odot \cdots$ and in $\wedge V$ we have $u = u_1 \wedge u_2 \wedge \cdots$ I'm assuming my vector spaces are finite dimensional. Define the following complex for V, d-dimensional:

   $$\to SV \otimes \wedge^d V \otimes SV \to \cdots \to SV \otimes V \otimes SV \to SV \otimes SV \to 0$$

   with an augmentation to SV.

   First of all I need to give you these maps b'. So

   $$b'(u \otimes v_1 \wedge \cdots \wedge v_r \otimes w) = \sum_{i=1}^{r} (-1)^{i+1} (u \odot v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_r \otimes w - u \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_r \otimes v_i \odot w)$$

   The augmentation is $\odot$. Note that if V is one dimensional then SV equals TV and this resolution is the same resolution we had before. If U and X are finite dimensional vector spaces then $P(U) \otimes P(X) \cong P(U \oplus X)$. So this is an isomorphism of dgas. It respects, you can go backwards. [Calculation] So we have an isomorphism of chain complexes and then we can put these together to get that this is a resolution for X of any dimension.

So now we can find out what our Hochschild is, tensoring over $SV \otimes SV^{op}$ with SV, and we get $wedgeV \otimes SV$. Let me do this more precisely: the maps $b' \otimes id$ transfer to $\wedge^* V \otimes SV$ as 0. Let me look at the diagram and show that it commutes. If we have $u \otimes v \otimes w$ and we tensor this with x. This maps to $v \otimes w \odot x \odot u$ which goes to zero. The other direction this will go to $\sum \pm(((u \odot v_i) \otimes \hat{v}^i \otimes w) \otimes x - (u \otimes \hat{v}^i \otimes (v_i \odot w)) \otimes x)$ which clearly goes to zero under this map.

So $HH_{.}(SV, SV) = H_{.}(P \otimes SV) = \wedge V \otimes SV$. As a homework, calculate $HH^{\cdot}(SV, SV)$.

I at least want to state the third example that one can do. The third example is to take $A = k[x]/x^{n+1} = 0$. The claim is that the following is a projective A^e resolution: $\cdots \to A^e \to A^e \to \cdots \to A^e$, where the augmentation is the product and you multiply by u and v alternatingly, where $u = x \otimes 1 - 1 \otimes x$ and $v = x^n \otimes 1 + x^{n-1} \otimes x + \cdots + x^n \otimes A \otimes A^{op}$. Then you can show that for every bimodule M over A, the $HH_{\cdot}(A, M)$ and $HH^{\cdot}(A, M)$ are two periodic and if $\frac{1}{n+1} \in k$ then $HH_i(A, A) \cong HH^j(A, A) \cong A/x^n A$

I want to continue with the coderivation point of view.

Let me start by saying what a derivation is. Let (A, μ) be an associative algebra. A derivation $D : A \to A$ is a linear map satisfying $D(a \cdot b) = Da \cdot b + (-1)^{|a||D|} a \cdot Db$. Here A is graded $A = \oplus A^i$ and the degree of a is $|a|$. (Assume μ is of degree zero).

The diagram I just wrote down was

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "D\otimes id + id \otimes D"'] & A \arrow[d, "D"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

The sign looks weird, but $(id \otimes D)(a \otimes b)$ is $(-1)^{|D||a|} a \otimes D(b)$.

A coalgebra is a pair (C, Δ) where C is a graded S-module and $\Delta : C \to C \otimes C$ such that we have coassociativity, degree zero. Coassociativity means that

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes id"] \\
C \otimes C \arrow[r, "id\otimes\Delta"'] & C \otimes C \otimes C
\end{tikzcd}$$

commutes.

A coderivation is a map $f : C \to C$ to make the following commute:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f\otimes id + id\otimes f"] \\
C \arrow[r, "\Delta"'] & C \otimes C
\end{tikzcd}$$

The main example we'll be using is the tensor coalgebra $T^c V$ which is TV with the structure

$$\Delta(v_1, \ldots, v_n) = 1 \otimes (v_1, \ldots, v_n) + v_1 \otimes (v_2, \ldots, v_n) + \cdots + (v_1, \ldots, v_n) \otimes 1$$

where $1 \in k = V^0$. Coassociativity corresponds to breaking things up in two ways, in either order.

You can characterize the coderivations on the tensor coalgebra nicely as follows. Every coderivation $TV \to TV$ is uniquely determined by its component $f^1 : TV \to TV \to V$.

There should be enough time to prove this. I don't want to write out the formula but in the end you will see it come out.

First of all, let $f^k : TV \to TV \to V^{\otimes k}$. We need to show that f^k is determined by f^1. We have the following, we know that f is a coderivation, so we have the following diagram:

$$\begin{tikzcd}
TV \arrow[r, "\Delta"] \arrow[d, "f"'] & TV \otimes TV \arrow[d, "f\otimes id + id \otimes f"'] \\
TV \arrow[r, "\Delta"'] & TV \otimes TV
\end{tikzcd}$$

If there was an f^0 then we'd have

$$\begin{tikzcd}
(v_1, \dots, v_m) \arrow[r, "\Delta"] \arrow[d] & 1 \otimes (v_1, \dots, v_m) + \dots + (v_1, \dots, v_m) \otimes 1 \arrow[dr] & \\
f^0(v_1, \dots, v_m)1_{TV} + \dots \arrow[r, "\Delta"'] & f^0(v_1, \dots, v_m)1 \otimes 1 & 2f^0(v_1, \dots, v_m)1 \otimes 1
\end{tikzcd}$$

These are different unless $0 = 1$ in k. If we assume that m is minimal in the sense that $f^0(v_1, \dots, v_k) = 0$ for $k \leq m$, then we get a contradiction that $1 = 2$.

So in the general case, $(v_1, \dots, v_m)$ goes to $\dots + f^k(v_1, \dots, v_m) + \cdots$ where this part will be called $(w_1, \dots, w_k)$ which splits in all possible ways, and lands in $V^{\otimes i} \otimes V^{\otimes k-i}$. For a given component $V^i \otimes V^{k-i}$, we can reconstruct f^k if we know what the value is on the other side. So by applying Δ we get, well, the only things that will land in $V^i \otimes V^{k-i}$ will be the things that have either i factors in the first part or $k - i$ in the second, and then what maps to use are determined by the factors they must land in.

So I get the statement that $f^k(v_1, \dots, v_m) = f^i(v_1, \dots, v_{m-k+i}, v_{m-k+i+1}, \dots v_m) + (-1)^{|f|(|v_1| + \dots + |v_i|}(v_1, \dots, v_i, f^k$