Deformation Theory and Operads

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Recall that we defined a formal 1 parameter deformation of an algebra (A, μ) as a $\mu \in C^1(A, A)$. Then $[\mu, \mu] = 0$ if and only if μ is associative. Then let (A, μ) be associative let $\mu' = \mu + t\tilde{\mu}$. Then μ' is a deformation of μ if and only if μ' is associative if and only if $[\mu', \mu'] = 0$ if and only if $2[\mu, t\tilde{\mu}] = 2\delta(t\mu) + [t\tilde{\mu}, t\tilde{\mu}] = 0$ so the Maurer Cartan equation for $t\mu$ or the master equation is

$$\delta(t\tilde{\mu}) + \frac{1}{2}[t\tilde{\mu}, t\tilde{\mu}] = 0$$

Let's do the Ext-Tor interpretation of Hochschild. Let R be a ring, let P be a left-module over R. Then P is called projective if it is a direct summand of a free R-module F so that there exists Q so that $F = P \oplus Q$. Let A be a free R-module; then a projective resolution of A is a chain complex $P_. = \to P_3 \to P_2 \to P_1 \to P_0$ so that $\epsilon P_0 \to A$, each P_i is projective, and so that $\ldots \to P_1 \to P_0 \to A \to \{0\}$ is an exact sequence.

For example, if A is projective, then $P_. = \ldots \to \{0\} \to \{0\} \to A = P_0$ is a projective resolution of A.

If A is an (left) R-module, and $P_.$ is a projective resolution of A, define the following:

Definition 1 *If B is a right R module, let $Tor_.^R(A, B)$ be the homology of $P_. \otimes_R B$*

One point is that these are independent of choices of B.

As an alternative, you look at the functor $(\otimes_R B) : R - mod \to Ab$, and say that this is a covariant right-exact functor and then define Tor^R is the left-derived of that functor.

The other one, Ext, is gotten by taking

Definition 2 *If B is a left R-module, then let $Ext_R(A, B)$ is the homology of $Hom(P_n, B)$ with the reversed differential. Alternatively, you could say $Hom_R(\ , B)$ is a contravariant left-exact functor from the category of R-modules into the category of Abelian groups, and Ext_R is the left derived of this.*

The main point is that Tor and Ext are independent of the choice of resolution.

I want to specialize to the case of Hochschild. Let's say (A, μ) be an associative algebra over some commutative ring S. What is the opposite algebra? Let (A^{op}, μ^{op}) be the opposite algebra. (This is ungraded). This means A^{op} is A and $\mu^{op}(a, b) = \mu(b, a)$. So switch the order of inputs; it's the opposite way of multiplying. Let (A^e, μ^e) be the enveloping algebra, so $A^e = (A \otimes A^{op}, \mu_e = \mu \otimes \mu^{op})$. We want to take $R = A^e$, so this will be our ground ring. What is Tor^{A^e} and Ext_{A^e}? So first, what is an A^e-module. If M is a left A^e-module, then we have maps $A^e \otimes M \to M$, so maps $A \otimes A^{op} \otimes M \to M$, so that's like $a \otimes b \otimes m \mapsto a.m.b$. This said, I think it's fairly easy to see that this has the module property. This is equivalent to M being a bimodule over A. Well, A is always a bimodule (right-A bimodule or left A-bimodule). So what is $Tor^{A^e}(A, M)$ or $Ext_{A^e}(A, M)$? We need a projective A^e-resolution of A. We can take $\cdots \to A^{\otimes 5} \to A^{\otimes 4} \to A^{\otimes 3} \to A^{\otimes 2}$ and you stop there.

So $A^{\otimes n \geq 2}$ is an A-bimodule, where you multiply on the left and on the right. It's projective, and a resolution of A as an A^e-module. Here the projection, on $(a_1 \to a_n)$, multiplies successive pairs togother, so is very similar to the Hochschild differential. So this is $(a_1 a_2) \otimes a_3 \otimes \cdots \otimes a_n + \cdots + (-1)^n a_1 \otimes \cdots \otimes \cdots (a_{n-1} \otimes a_n)$ This is a differential. We haven't discussed the Hochschild differential. ϵ is the product in A.

Now to prove this, why is P projective? If A is projetive over S and A has a left unit, well, [discussion]. Why is the sequence exact? What you do is define $s : A^{\otimes n} \to A^{\otimes n+1}$ which takes $a_1 \otimes \cdots \otimes a_n \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_n$. Then $b' \circ s + s \circ b' = id_{A^{\otimes n}}$, where b' is the projection map (including augmentation). To calculate we see

$$b' \circ s(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots a_n - 1 \otimes (a_1 a_2) \otimes \cdots a_n + \cdots \pm 1 \otimes a_1 \otimes \cdots \otimes (a_{n-1})(a_n)$$

On the other hand, $s \circ b'$ is the same thing without the first term. So some of the terms cancel and you just get the leading term of the first. This is a typical calculation.

Now we can just plug this in. Let M be a bimodule over A. Then $Tor^{A^e}(A, M) = H_.(P_. * \otimes_{A^e} M)$ is the homology of $A^{\otimes 4} \otimes_M \to A^{\otimes 3} \otimes M \to A^{\otimes 2} \otimes M \cong M$

What is the differential? We get for example $b'(a_1 \otimes a_2 \otimes m)$, which can be written $(1 \otimes a_2 \otimes a_3 \otimes 1) \otimes_{A^e} m$ which is $a_2 \otimes a_3 \otimes s \otimes m - (1 \otimes a_2 a_3 \otimes 1) \otimes m + (1 \otimes a_2 \otimes m_3) \otimes m$. So under the identification this gives $a_3 \otimes m a_2 - (a_2 a_3) \otimes m + a_2 \otimes (a_3 m)$.

In general I get the Hochschild differential. You multiply things together and get $a_2 \otimes \cdots \otimes (m a_1) - a_1 a_2 \otimes \cdots \otimes m$ and so on. $Ext_{A^e}(A, M)$ has exactly the Hochschild differential.

There are a few easy things. So let $A = TV = \bigoplus V^{\otimes}$ be the tensor algebra with product the tensor product. Then by this lemma, we can choose any projective resolution. This one has a small resolution, $P_.^{sm}$ is $TV \otimes V \otimes TV \to TV \otimes TV$. This is a resolution, with the only map b' equal to

$$b'((v_1, \ldots, v_n) \otimes v \otimes (w_1, \ldots, w_m)) := (v_1, \ldots, v_n, v) \otimes (w_1, \ldots, w_m) - (v_1 \ldots, v_n) \otimes v, w_1, \ldots, w_n)$$

I claim this is a projective resolution of TV. What you get is that $Tor^{TV^e}(TV, TV) = H_.((TV \otimes V) \to TV)$ which is TV_τ (as a homework), the coinvariants under the cyclic rotation, so $TV/1 - \tau$, in $i = 0$ and for $i = 1$ the elements $v \in TV$ so that $\tau(v) = v$.

One has to sit down and do these calculations, but you can just sit down and do them, it's totally spelled out, it's doable. The second example using this is $A = SV$. We can do the same thing, and again we need a resolution. Note that if V is 1-dimensional, then $TV = SV$, so wo can use the first case. If this is not one-dimensional, assume it is finite dimensional. You want to build the resolution out of these for , well, we get $P^{SV} - SV \otimes \wedge V \otimes SV$. So $P^{sm}_{\cdot}$ has $SV \otimes \wedge^d V \otimes SV \to SV \otimes \wedge^{d+1} V \otimes SV \to \cdots$ Now you do the same thing as before and you get $(u_1, \ldots, u_n) \otimes (v_1 \wedge \cdots \wedge v_m) \otimes (w_1, \ldots, w_k)$ is the sum of bringing one v_i into the left and right factor in all possible ways with the appropriate signs.

I will say some more next time, some resolution, and then some examples. The homology is $HH_i(SV, SV)$ is $SV \otimes \wedge^i V$ for $0 \leq i \leq d$. All the differentials become zero. I think I'll do more about this next time and the coderivation. This example where the algebra is SV is like functions on the manifold. So the Hochschild on functions on a manifold is the smooth forms.