Deformation Theory and Operads

Gabriel C. Drummond-Cole

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Let me go over a couple of things from last time.

The example of a point makes sense, because it's really a bicomplex. d stays within one A and then id or 0, that's the same as in the other order.

When I stated the theorem about the Hochschild complex, the Hochschild homology of chains $H_\cdot(C^{X_\cdot}(M)) \cong H^\cdot(Map(|X|, M))$, and I didn't prove that this was a quasiisomorphism. To have the spectral sequence collapse, you need M to be n-connected at least, where n is the dimension of the highest nondegenerate simplex in X.

Finally, I do want to talk about operads. Basically this whole operad part is a generalization of all of these ways of viewing the Hochschild complex.

I'm going to do a lot of examples today. Most of them you will know, but basically I want to state some of these old theorems from the 70s. There are two big books that used these. Bordmann, Vogt and May were the two books. They were dealing with iterated loop spaces. My goal is to prove something in one of the easiest cases.

Before I do the definition, let me give motivation. For today I want to stay just with the two categories, $\mathscr{C}$ the category of topological spaces, or let $\mathscr{C}$ be the category of vector spaces (today ungraded) over a field k. Next time I'll do this more generally, for monoidal categories. Let's actually say we're doing the category of vector spaces over a field k. I can look at $End_V(n) = Hom(V^{\otimes n}, V)$, for $n \geq 1$ and V a vector space. An operad is an abstract version of saying what a structure on these spaces is. Say $f \in End_V(n)$ and $g \in End_V(m)$, then we can make a circle product $f \circ_i g \in End_V(n+m-1)$ and $f \circ_i g(v_1, \ldots, v_{n+m-1}) = f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+m-1}), v_{i+m}, \ldots, v_{n+m-1})$. This is exactly the picture of a tree sitting on top of another tree. There is a right Σ_n-action on $End_V(n)$. $(f.\sigma)(v_1, \ldots, v_n) = f(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}$.

[Subsection on group actions. $(v_1, \ldots, v_n).\sigma = v_{\sigma(1)}, \ldots, v_{\sigma(n)}$ is a right action; $\sigma.(v_1, \ldots, v_n) = v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}$ is a left action.]

There is also one privileged element in $End_V(1)$, the identity $\mathbf{1} : V \to V$.

Definition 1 *Let $\mathscr{C}$ be Top or Vect, then an operad in $\mathscr{C}$ consists of a sequence of spaces*

$\{\mathscr{O}(n)\}_{n\geq 1}$ *with* $\mathscr{O}(n)$ *in Ob* $\mathscr{C}$ *and*

a *For* $n, m \geq 1$, $1 \leq i \leq m$ *there is a morphism* $\circ_i : \mathscr{O}(m) \otimes \mathscr{O}(n) \to \mathscr{O}(m+n-1)$ *such that we have the following associativity axiom for* $f \in \mathscr{O}(m)$, $g \in \mathscr{O}(n)$, *and* $h \in \mathscr{O}(p)$*:*

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+p-1} g & 1 \leq j \leq i-1 \\ f \circ_i (g \circ_{j-i+1} h) & i \leq j \leq i+n-1 \\ (f \circ_{j-i+1} h) \circ_i g & i+n \leq j \leq n+m-1 \end{cases}$$

b *For* $n \geq 1$, *there is a* Σ_n*-right action on* $\mathscr{O}(n)$ *such that we have the "equivariance axiom" stating that for* $f \in \mathscr{O}(m)$, $g \in \mathscr{O}(n)$

$$(f.\sigma) \circ_i (g.\rho) = f \circ_{\sigma(i)} g).(\sigma \circ_i \rho)$$

I need to explain what these symbols mean, that is, $\sigma \circ_i \rho$. *This is*

$$\sigma_{1,\dots,\underbrace{n}_{i},\dots,1} \circ 1 \times \cdots \times \underbrace{\rho}_{i} \times \cdots, 1)$$

So

$$1 \times \cdots \times \underbrace{\rho}_{i} \times \cdots \times 1)(j) = \begin{cases} j & j < i, j \geq i+n \\ \rho(j-i+1)+i-1 & \textit{otherwise} \end{cases}$$

and

$$\sigma_{1,\dots,\underbrace{n}_{i}1\dots,1}(j) = \begin{cases} \sigma(j) & j \leq i-1 & \sigma(j) < \sigma(i) \\ \sigma(j)+n-1 & j \leq i-1 & \sigma(j) > \sigma(i) \\ \sigma(i)+j-i & 1 \leq j \leq i+n-1 & \\ \sigma(j-n+1) & j \geq i+n & \sigma(j-n+1) < \sigma(i) \\ \sigma(j-n+1)+n-1 & j \geq i+n & \sigma(j-n+1) > \sigma(i) \end{cases}$$

c *There exists a special element* $\mathbf{1} \in \mathscr{O}(1)$ *such that we have the unit axiom:*

$$f \circ_i \mathbf{1} = f; \mathbf{1} \circ_1 f = f$$

Examples.

1. If V is a vector space then $End_V(n)$ is an operad, called the endomorphism operad. If X is a topological space, $End_X(n) = Hom_{cont}(X^{\times n}, X)$ is an operad called the endomorphism operad.

2. Another example, let k be a field, let's say characteristic 0. Then $Com(n) = k$ for all $n \geq 1$. Define the $\circ_i : k \otimes k \to k$ as multiplication in k, the Σ_n action is the identity, and the unit is the unit of the field. This defines an operad, the commutative operad.

This is pretty trivial, everything is the identity or multiplication. Everything is pretty canonical. Next time I'll say why it deserves this name.