Deformation Theory and Operads

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July 9, 2012

Recall, we start with a finite simplicial set $X_\cdot : \Delta \to Set_f$ and with $(A, d_A, \cdot)$ a graded commutative associative unital algebra. (We can add basepoints if these are pointed sets.

Then The Hochschild chain copmlex $C_\cdot^{X_\cdot}(A)$ is a simplicial chain complex given as follows. $\bar{n}$ is mapped to $C_\cdot^{X_n}(A)$ which by definition is $A^{\otimes X_n}[n]$. This has the differential induced by d_A. Let me point out that for a map $f : X_k \to X_\ell$ we have induced maps on the chain complexes $A^{\otimes X_k} \to A^{\otimes X_\ell}$. Let me use x_k as $|X_k|$. Then

$$b_j = \prod_{i \in X_k : f(i) = j} a_i$$

An empty product is the unit. So I induce a map $f_\# : A^{\otimes X_k}[k] \to A^{\otimes X_\ell}[\ell]$ which is now a map of degree $k - \ell$.

I told you where all these maps go to. The most important are the maps d, the boundary maps $d_i : X_n \to X_{n-1}$, and w get maps $(d_i)_\# : C_\cdot^{X_n}(A) \to C_\cdot^{X_{n-1}}(A)$ of degree 1.

That's pretty much it. You get the induced map of degree plus one and the total Hochschild cochain complex is defined to be

$$C_\cdot^{X_\cdot}(A) = \bigoplus_{n \geq 0} C_\cdot^{X_n}(A)$$

with differential $\delta = \tilde{d}_A + \sum_{i=0}^{n}(-1)^i (d_i)_\#$ which is degree one and squares to zero.

Each piece in this sum squares to zero, and these two things commute, so $\delta^2 = 0$. We can see that $(d_i)_\#$ squares to zero: When we square we get

$$(\sum_{i=0}^{n}(-1)^i(d_i)_\#)(\sum_{j=0}^{n-1}(-1)^j(d_j)_\#) = \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}\,(d_i)_\#\circ(d_j)_\# = \sum_{0\leq i\leq j\leq n-1}(-1)^{i+j}(d_id_j)\#+\sum_{0\leq j<i\leq n}(-1)^{i+j})(d_i$$

This can be renamed, so let $r = i - 1$ and we have this second sum as

$$\sum_{0\leq j\leq r\leq n-1}(-1)^{r+j+1}(d_{r+1}\circ d_j)_\#$$

I want some examples. Remember S^1_n was $\{0, \dots, n\}$. Then $C_\cdot^{S^1_n}(A) = A^{\otimes n+1}[n]$. Then d_i maps i and $i+1$ to i. Then the last one d_n takes n to 0. This induces $(d_i)_\#(a_0, \dots, a_n)$ which multiplies $(a_0, \dots, a_i a_{i+1}, \dots, a_n)$ and $(d_n)_\#(a_0, \dots, a_n) = (a_n a_0, \dots, a_{n-1})$. Let me remind you of the degeneracy maps take $(s_i)_\#(a_0, \dots, a_n) = (a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n)$. In general, you can mod out by degeneracies, they give an acyclic subcomplex.

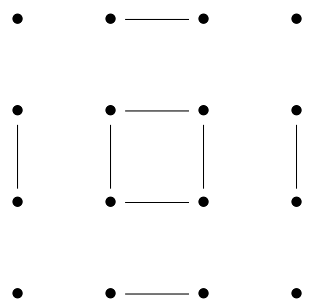
Let's do the point. $*_n = \{0\}$, it's a point in each degree, and $C_\cdot^{*_n}(A) = A^{\otimes 1} = A$. The boundaries are identity maps, and the s_i are all also just the identity. The differential is then $id + d_A$ or d_A,

For the interval, $I_n = \{0, \dots, n+1\}$, and $C_\cdot^{I_n}(A) = A^{\otimes n+2}[n]$ and you get $C_\cdot^{I_\cdot}(A) = A^{\otimes n+2} \to A^{\otimes n+1} \to \cdots$ The differential d_i maps i and $i+1$ to i so $(d_i)_\#(a_0, \dots, a_{n+1}) = (a_0, \dots, a_i a_{i+1}, \dots, a_{n+1})$. This ends with $A^{\otimes 2}$.

I wanted to apply this to de Rham forms. Let's do another example, the torus. These are tuples (i, j) with these between 0 and n. This is really $S^1_\cdot \times S^1_\cdot$. The chains are $C_\cdot^{T_n}(A) = A^{(n+1)^2}[n]$. The differential multiplies in a particular way:

$$(d_2)_\# \begin{pmatrix} a_1 & a_2 & \cdots & a_5 \\ a6 & \cdots & & \\ & & \ddots & \\ & & & a_{25} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 a_4 & a_5 & \\ a_6 & a_7 & a_8 a_9 & a_{10} & \\ a_{11} a_{16} & a_{12} a_{17} & a_{13} a_{14} a_{18} a_{19} & a_{15} a_{20} & \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}$$

So one more example, the sphere, recall $S^2_\cdot$ which had $S^2_n = (i, j) : 1 \le i, j \le n$ and $(0, 0)$. Then $d_2 : S^2_4 \to S^2_3$ obeys the following (ignoring the $(0, 0)$)



That one's too easy, lets do d_0:

$$(d_0)_\# \begin{pmatrix} a_0 & & & & \\ & & & & \\ & a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{21} & a_{22} & a_{23} & a_{24} \\ & a_{31} & a_{32} & a_{33} & a_{34} \\ & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_0 a_{11} a_{12} a_{13} a_{14} a_{21} a_{31} a_{41} & & & \\ & & & \\ & a_{22} & a_{23} & a_{24} \\ & a_{32} & a_{33} & a_{34} \\ & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

This is a fun calculation. Work it out for the higher dimensional torus too.

Let me make a remark. This is functorial both in the simplicial sense $X_. \to Y_.$ and also in the induced map $A \to B$. This also respects limits so you can pass at least to countable sets.

Let M be a manifold and let $X_.$ be a finite simplicial set. Denote by $X = |X_.|$. Denote by ΩM the de Rham forms of M with differential d_{DR} and product $\wedge$. We also want to let M^X be the mapping space $f : X \to M$ which are continuous with the compact open topology.

Then $C^{X\cdot}_{.}(\Omega M)$, we can take its homology, and can take $H^{\cdot}(M^X)$ and these are isomorphic. A special case of this is where $X = S^1_.$, and then $HH(\Omega M)) \cong H^{\cdot}(LM)$, the loop space on M. I will construct a chain level map and this was Chen's iterated integral map. The various versions of that, Dennis had his fingers in there. Chen quotes Dennis at some point. Someone else is given credit for the S^1 version sometimes, McClure or someone?

[Reference: Chen, Annals 73, Iterated integrals of differential forms and loop space homology]

Let me construct a simplified version of this, a map from $C^{X\cdot}_{.}(\Omega M) \to C^{\cdot}(M^X)$, singular simplicial cochains of M^X. I'll call this map *Chen*. We need to define it for an element $\alpha \in C^{X\cdot}_{.}(\Omega M)$. So if σ is a chain, I should be able to say $Chen(\alpha)(\sigma) \in k$. This will define the map.

Assume that σ is a single simplex $D \to M^X$. Let $\rho_\sigma : \Delta^K \times D \to \Delta^K \times M^X \to M^{X_k}$. The first map is $id \times \sigma$. Now the second one is into maps from the set X_k to M. So this is just $M^{|X_k|}$. This map is evaluation. It goes from $\Delta^k \times M^X \to M^{X_k}$ via, well, recall that $|X_.| = \amalg X_k \times \Delta^k / \sim$. So given a point in Δ^k and a map: $X_k \times \Delta^k \to M$ gives a map from $X^k \to M$.

So $Chen(\alpha)(\sigma)$, well, $(\alpha \in \Omega M)^{X_k}$, we can take $\rho^*_\sigma(\alpha)$ which is a form on $\Delta^k \times D$. You integrate over $\Delta^k \times D$ and that gives you a number. I guess I'm over time. One last thing, let me indicate why this is a chain map.

$$(\partial Chen(\alpha))(\sigma) = Chen(\alpha)(\delta\sigma) = \int_{\Delta^k \times \delta D} (\rho_{\sigma|_{\delta D}})^*(\alpha) = \pm \int_{\delta(\Delta_k \times D)} (\rho_\sigma)^*(\alpha) \pm \int_{\delta\Delta^k \times D} (\rho_\sigma)^*(\alpha) = \pm \underbrace{\int_{\Delta^k \times D} d_{DR}}_{\text{``}\tilde{d}_A\text{'' part of the}}$$

and then looking at this diagram we can see

$$\begin{tikzcd}
\delta\Delta^k \times D \arrow[r, "id\times\sigma"] & \delta\Delta^k \times M^X \arrow[r, "eval"] & M^{X_k} \\
\amalg_{i=0}^k \Delta^{k-1} \times D \arrow[u, equal] \arrow[r, "id\times\sigma"'] & \amalg_i \Delta^{k-1} \times M^X \arrow[r, "eval"'] & \amalg M^{X_{k-1}} \arrow[u, "(d_i)_\#"']
\end{tikzcd}$$

so

$$\int_{\delta\Delta^k \times D} (\rho_\sigma)^*(\alpha) = \sum_{i=0}^n \pm \int_{\Delta^{k-1} \times D} (\rho_\sigma)^*((d_i)_\#(\alpha)) = \sum_{i=0}^n \pm Chen((d_i)_\#(\alpha))(\sigma)$$

This map is a quasiisomorphism as can be seen from the Bowfield Kan spectral sequonce, in American Journal of Math, 1987, number 109. "The homology spectral sequence of a cosimplicial space."