Deformation Theory and Operads

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July 9, 2012

Recall that $\mathbf{\Delta} \to \mathscr{C}$ is a cosimplicial or simplicial object in $\mathscr{C}$, corresponding to whether the functor is covariant or contravariant. $\Delta^\cdot$ is a cosimplicial topological space and for X in Top, $S(X)$ is a simplicial set.

Let me give you a small example. $S^1_\cdot : \mathbf{\Delta} \to Set$ is a simplicial set such that $|S^1_\cdot| = S^1$. We need S^1_n, which are sets, along with face and boudary maps $s_j : S^1_n \to S^1_{n+1}$ for $j = 0, \ldots, n$ and $d_j : S^1_n \to S^1_{n-1}$ over the same range.

Let S^1_n be the set $\{0, \ldots, n\}$. The map s_j is the identity up to j and then afterward it adds one.

Those are the degeneracy maps. I'm also going to give you the d_j, which are of two kinds. d_n is different than the other ones. The first n map j and $j+1$ to j and then above $j+1$ subtract one. Then d_n is the identity but maps n to 0.

I claim that this is a simplicial set. I don't want to prove this; to do this, use the lemma. What are the nondegenerate simplices? These do not come as the image of s. So we have

$$\begin{tikzcd}
S^1_0 \arrow[r, "s_0"] & S^1_1 & S^1_2 & \\
\underbrace{0}\text{ nondeg} \arrow[r] & 0 \arrow[r, bend left, "s_0"] \arrow[r, bend right, "s_1"'] & 0 & \\
& \underbrace{1}\text{ nondeg} \arrow[r, bend left, "s_1"] \arrow[r, bend right, "s_0"'] & 0 & 2
\end{tikzcd}$$

Therefore the geometric realization is just the disjoint union moduo an equivalence. So $j, 0 \le t_1 \le \cdots \le t_n \le 1]$ is the j^{th} n-simplex in the geometric realization.

So if I take this tuple, 2 is $s_0(1)$ so under equivalence this is the same as $(1, S_0(0 \le t_1 \le t_2 \le 1$

So all of these except the 0 in each S^1_n is associated to $(1, 0, \le t_2 \le \cdots 1)$.

Both of the boundaries of the 1 in S^1_1 should be identified with the right and left boundaries on the point.

Maybe one thing to point out is that this will give the Hochschild complex, because how do you do the boundary? You multiply a bunch of things. I will make this precise in a moment.

Nothing prevents us from looking at another one, the torus. $T_n = S^1Sn \times S^1_m$. You apply the boundaries pointwise, and degeneracies as well.

Let me do the lowest terms and see what are the degenerate objects. I have T_0, T_1, and T_2, which have two, four, and eight bits in them. So there are three nondegenerate edges corresponding to $(0,1)$, $(1,0)$, and $(1,1)$. There are two nondegenerate 2 simplices. All the higher ones are degenerate. If you put these together, you get the following pictur: [picture]

If you look at, say, some boundary maps $T_4 \to T_3$, you get a big five by five matrix mapping down to a four by four.

The boundary is just component by component. [pictures]

[example of the simplicial sphere.]

In general, if you have a surface with v vertices, e edges, and f faces, then Σ_n has $\binom{n}{0}v + \binom{n}{1}e + \binom{n}{2}f$.

Let me stop with this for a moment.

Let's go back to the Hochschild complex. The idea is that if A is a differential graded commutative algebra, then there is a functor from finite sets to differential graded modules, which takes a finite set $\{0,\dots,n\}$ to $A^{\otimes n+1}$. For a finite set Z we can define the unordered tensor product. $A^{\otimes Z}$ is a sum over bijections $Z \to \{1,\dots,n\}A \otimes \dots A$ modulo some equivalences.

So $0,\dots,\underbrace{a_1 \otimes \cdots a_n}_{f},\dots,0$ is eqivalent to $0\dots,a_{\sigma(1)} \otimes \cdots a_{\sigma(n)_{\sigma\circ f}},\dots,0$.

Later I will plug in the de Rham forms on a manifold.

Definition 1 *Let A be a dg associative commutative unital algebra. Let $X_. : \mathbf{\Delta} \to Set_f$ be a simplicial set. (I don't think I need finiteness). Then the Hochschild chain complex of A with respect to $X_.$ is the simplicial module (differential graded module) $C^{X_.}_.(A) : \mathbf{\Delta} \to dg - Mod$ which is just this composition. This is a simplicial dg-module. It is bigraded, $C^{X_n}_.(A)$ is a dg-module over k defined by $C^{X_n}_.(A) = A^{X_n}[n]$, shifted down by n. The differential is the tensor product of the differentials of A. Just to have said it once, for the simplicial maps, these come from the definition that for a map $f : X_k \to X_\ell$ we have an induced map $f_\# : A^{\otimes X_k} \to A^{\otimes X_\ell}$. This takes $a_1,\dots,a_{X_k}$ to $b_1,\dots,b_{X_\ell}$ where b_j is $\prod_{i\in f^{-1}(j)} a_i$ or 1 if the preimage is empty.*

I'll make a couple of examples applying this to the circle and torus next time, and assembling this to a total complex. This is the main point, that you look at the preimage of your main point.