Deformation Theory and Operads

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Let me give the simplicial interpretation of the Hochschild complex. Let's start with a prelude on simplicial objects.

Definition 1 *Let Δ be the category with objects $\bar{n} = \{0, \ldots, n\}$ for $n \geq 0$ and morphisms ar maps $f : \{0, \ldots, n\} \to \{0, \ldots, n\}$, non-decreasing. Composition is composition of set maps.*

The most important maps are the maps $\delta_i : \bar{n} \to n \bar{+} 1$ which skips i and is injective. Let $\sigma_i : \bar{n} \to n \bar{-} 1$ be the surjective map that hits i twice.

As homework, every non-decreasing map $f : \bar{n} \to \bar{m}$ can be written as $f = \delta_{i_1} \circ \cdots \delta_{i_r} \circ \sigma_{j_1} \circ \cdots \sigma_{j_s}$.

This is unique if $i_r \leq \cdots i_1$ and $j_s \leq \cdots \leq j_1$. We have the relations $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$ and $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$ along with $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ for $i < j$, $id_{\bar{n}}$ for $i = j$ or $i = j + 1$, and $\delta_{i-1} \sigma_j$ for $i > j + 1$

Definition 2 *Let $\mathscr{C}$ be a category. A cosimplicial object in $\mathscr{C}$ is a covariant functor $X^\cdot : \Delta \to \mathscr{C}$ A simplicial object in $\mathscr{C}$ is a contravariant functor $X_\cdot : \Delta \to \mathscr{C}$*

Lemma 1 *a A cosimplicial object in $\mathscr{C}$ is equivalent to a sequence of objects $X^i \in Obj(\mathscr{C})$ and morphisms in $\mathscr{C}$ $D_i : X^n \to X^{n+1}$ for $0 \leq i \leq n+1$, $S_i : X^n \to X^{n-1}$ for $0 \leq i \leq n-1$ satisfying the corresponding commutation relations.*

b A simplicial object is a sequence of objects $X_0, X_1, \ldots$ and morphisms $d_i : X_{n+1} \to X_n$ for $0 \leq i \leq n+1$ ando $s_i : X_{n-1} \to X_n$ for $0 \leq i \leq n-1$ satisfying $d_i d_j = d_{j-1} d_i$ for $i < j$, $s_i s_j = s_{j+1} s_i$ for $i \leq j$ and then

$$d_i s_j = \begin{cases} s_{j-1} d_i & , \quad i < j \\ id_{X_n} & , \quad i = j, i = j+1 \\ s_j d_{i-1} & , \quad i > j+1 \end{cases}$$

The proof for the first part is, take $X^n = X^\cdot(\bar{n})$ and $D_i = X^\cdot(\delta_i)$,$S_i = X^\cdot(\sigma_i)$. This works both forward and backward. The proof of the second is the same.

The most common example is to have $\mathscr{C}$ be topological spaces and then let $\Delta^n = \{(t_1, \dots, t_n)\}|0 \leq t_1 \leq \dots \leq 1$

This can be made into a cosimplicial topological space: $\Delta^{\cdot} : \not\Delta \to Top$ takes Δ^n as above and $D_i : \Delta^n \to \Delta^{n+1}$ which maps $t_1 \dots, t_n$ to $t_1, \dots, t_i, t_i, \dots, t_n$ and S_i which maps $t_1, \dots, t_n \mapsto t_1\, ldots, \hat{t}_i, \dots, t_n$

Let X be a topological space. It has no bullet so it's not simplicial or cosimplicial. The singular n-simplices of X are $S(X)_n := \{r : \Delta^n \to X | r \text{ continuous}\}$. These can be put together to give a simplicial set.

This cosimplicial thing can be composed with the contravariant functor $Hom(\bullet, X)$ to give this as a contravariant functor $\not\Delta \to Set$. This says $d_i(r) = r \circ D_i$ and $s_i(r) = r \circ S_i$. So $S(X)$ is a functor from topological spaces to simplicial sets.

I want to say what the geometric realization is, which is a functor in the opposite direction. It comes up all over the place. We have a geometric realization functor $T = |\quad|$. This takes, for a simplicial set $X_{\cdot}$ it takes

$$T(X_{\cdot}) = |X_{\cdot}| = \amalg_{n \geq 0} X_n \times \Delta^n / \sim$$

Where the equivalence relation says $(d_i(x), t_1, \dots, t_n) \sim (x, D_i(t_1, \dots, t_n))$ and $(s_i(x), t_1, \dots, t_n) \sim (x, S_i(t_1, \dots, t_n))$.

These are adjoint functors so there is a bijection between $Hom_{Top}(|X_{\cdot}|, Y) \to Hom_{Simp\ Set}(X_{\cdot}, S(Y)_{\cdot})$ which is almost tautological. I start with $f : \amalg X_n \times \Delta^n / \sim \to Y$. I want to give a bunch of maps $f_n : X_n \to Hom(\Delta^n, Y)$. If you look at this this is almost immediate. You define $f_n(x)(t_1, \dots, t_n)$ as $f((x, t_1, \dots, t_n))$.

If $X_{\cdot}$ is a simplicial set then $x \in X_n$ is called degenerate if $x = s_i(y)$ for some y and i. Otherwise it is nondegenerate.

I want to do one more different example, a simplicial set that gives the circle. Next time I'll do more examples. So the simplicial circle $S^1_{\cdot}$ The goal is to define $S^1_0, \dots, S^1_i$ which should give the simplices so that the geometric realization is the circle. So $S^1_0 = \{0\}$. Then S^1_1 has $s_0(0)$ so I need at least one more simplex, one that is nondegenerate. I can let the set be $\{0, 1\}$. I want $d_i s_i = id_{X_n}$. So s_i must be injective. I don't know how much I want to go through this, but it turns out the smallest thing you can do is the following:

S^1_0	0
S^1_1	$0, 1$
S^1_2	$0, 1, 2$
S^1_3	$0, 1, 2, 3$
S^1_4	$0, 1, 2, 3, 4$

If you analyze the s_i and d_i you get that $s_i : S^1_n \to S^1_{n+1}$ for $0 \leq i \leq n$ is given by $s_i(k)$ is either k for $k \leq i$ or $k + 1$ for $k > i$.

The d_i are given by $d_i(k) = k$ if $k \leq i$ and $k - 1$ if $k > i$. This gives all of them but $d_n(k)$ which is k if $k \leq n - 1$ or 0 if $k = n$. So d_5 is not a nondecreasing map. I'm already fifteen minutes late. I want to show that the geometric realization has just the two nondegenerate simplices.