Deformation Theory and Operads

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Let $(A, D = \mu_1 + \mu_2 + \cdots)$ be an A_∞ algebra with $\mu_1 =: d_A$, and (B, d_B) a chain complex. Let $f : A \to B$ and $g : B \to A$ be chain maps, and $h : A \to A$ be a homotopy: $hd_A + d_B h = gf - id_A$. Then there exists

- An A_∞ structure (B, ν) extending $\nu_1 = d_B$
- an A_∞ algebra map $F : TA[1] \to TB[1]$ with $F^1 = f$.
- an A_∞ algebra map $G : TB[1] \to TA[1]$ with $G^1 = g$.
- an A_∞ homotopy (I don't want to talk about this, really) $H : TA[1] \to TA[1]$ extending h satisfying $HD + DH = GF - id_{TA[1]}$.

I used the following notation. Did I use blue or red for A? [Pictures] You introduce three new terms when dragging. You get terms with fg, terms with h and a decomposition, and terms with the identity.

Definition 1 *A Hodge decomposition of a chain complex (V, d_V) is a decomposition $V = W \oplus B \oplus R$ such that $B = Im(d_V)$ and $W \oplus B = \ker(d_V)$. Hdet $H_.(V, d_V) = \frac{W \oplus B}{B} \cong W$ and $d_V(W \oplus B) = \{0\}$. Then $d_V|_R : R \to B$ is an isomorphism since $B = Imd_V$ and $R \cap W \oplus B = \{0\}$.*

Note also that $f : W \oplus B \oplus R \to W$ and the reverse inclusion are homotopy inverses: $fg - id_W$ is 0 and $gf - id_V$ is the projection onto $B \oplus R$. It's not hard to see that this is $-dd^{-1} - d^{-1}d$. Then every finite dimensional chain complex has a Hodge decomposition.

Corollary 1 *Kadeishvili 82 (The algebra structure on the homology of an A_∞ algebra Let (A, D) be an A_∞ algebra and let H be the homology of A with respect to μ_1. Then there exists an A_∞ structure on H with $\nu_1 = 0$ and A_∞ algebra maps $F : TA[1] \to TH[1]$ and $G : TH[1] \to TA[1]$ and a homotopy H with $GF - id = HD + DH$.*

This comes from choosIng a Hodge decomposition and applying the transfer theorem. For example, ν_n is the sum over all kinds of trees where you put things from H to V, use products in V, intersperse with homologies d^{-1}, and eventually project back.